Universal Gröbner bases for maximal minors

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to appear in International Mathematics Research Notices

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- $\{g_1, \ldots, g_r\}$ is a Gröbner basis of I wrt $\prec$ if

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- $\{g_1, \dots, g_r\}$ is a universal Gröbner basis of I if it is a Gröbner basis of I with respect to any term order on S, that is

$$\mathrm{in}_\prec(I) = (\mathrm{in}_\prec(g_1), \dots, \mathrm{in}_\prec(g_r)) \text{ wrt every } \prec$$

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Variants: generic symmetric matrix, generic skew-symmetric matrix (and ideals of pfaffians), generic Hankel matrices
They appear in various contexts, e.g.

- classical invariant theory,
- $t = 2$: defining ideal of the Segre/Veronese/Grassmannian variety,
- higher t: secant varieties of Segre/Veronese/Grassmannian variety.

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Similar results for skew-symmetric, symmetric, Hankel matrices, powers and product of determinantal ideals

Universal GB for minors, $t = 2$

Theorem (Sturmfels, Villareal)

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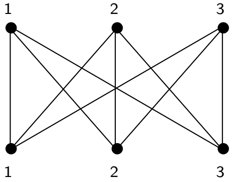
$K_{3,3}$:



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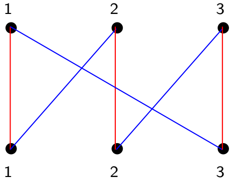

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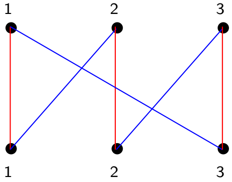

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All the initial ideals of $I_2(X)$ are radical and define CM rings (indeed they are associated to a shellable simplicial complex)

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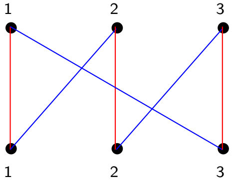

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$X = (x_{ij})$ generic matrix of size $m \times n$, $m \leq n$.

a minor of size m is called maximal minor: $[c_1, \ldots, c_m]_X$

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Boocher (2011)

For every term order $\prec$:

- $\beta_{ij}(I_m(X)) = \beta_{ij}(\mathrm{in}_\prec(I_m(X)))$
- in particular $\mathrm{in}_\prec(I_m(X))$ has a linear resolution

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Lemma

Let M, T be f.g. graded S-modules and let $J = (z_1, \ldots, z_s) \subset S$ be a homogeneous ideal. Suppose that:

(1) $\mathrm{HS}(T, y) \geq \mathrm{HS}(M, y)$ coefficentwise.

(2) $\mathrm{HS}(T/JT, y) = \mathrm{HS}(M/JM, y)$

(3) $z_1, \ldots, z_s$ is M-regular sequence.

Then $\mathrm{HS}(T, y) = \mathrm{HS}(M, y)$ and $z_1, \ldots, z_s$ is a T-regular sequence.

Universal GB for maximal minors: "short proof"

Fix $\prec$ any term order on S.
$D = (\mathrm{in}_{\prec}([c_1, \ldots, c_m]_X) : 1 \leq c_1 < \ldots c_m \leq n) \subseteq \mathrm{in}_{\prec}(I_m(X))$

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Pick $A = (a_{ij}) \in M_{mn}(K^*)$ such that all max minors are $\neq 0$ and new variables $y_1, \dots, y_n$.

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Mapping x_{ij} to $a_{ij} y_j$, we get a K-algebra map

$$\Phi : S = K[x_{ij}] \longrightarrow K[y] = K[y_1, \ldots, y_n]$$

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- m monomial of $[c_1, \dots, c_m]_X \Rightarrow \Phi(m) = \alpha y_{c_1} \cdots y_{c_m}$, $\alpha \in K^*$

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J gen. by M and T-reg. seq. $\Rightarrow \beta_{ij}(I_m(X)) = \beta_{ij}(\mathrm{in}_{\prec}(I_m(X)))$

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Is it possible to prove similar results for matrices of linear forms?

Let $L = (L_{ij})$ an $m \times n$ matrix, $m \leq n$, with $L_{ij} \in R_1$.

Eagon-Northcott

$\text{height}(I_m(L)) \leq \text{height}(I_m(X)) = n - m + 1$
If $=$ holds, then the Eagon-Northcott complex is a minimal free resolution of $I_m(L)$

What can go wrong

Example 1

$$L = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & 0 & x_4 \end{pmatrix} \qquad m = 2, n = 4, \mathrm{height}(I_2(L)) = 2 < 3$$

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Example 2

$$L = \begin{pmatrix} x_1 + x_2 & x_3 & x_3 \\ 0 & x_1 & x_2 \end{pmatrix} \quad m = 2, n = 3, \operatorname{height}(I_2(L)) = 2$$

$\mathrm{in}_{\prec}(I_2(L))$ has a generator in degree 3 for every $\prec$ (if $\mathrm{char}(K) \neq 2$)
$\Longrightarrow$ $I_2(L)$ has no GB of quadrics

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$$L = \begin{pmatrix} x_1 & x_4 & x_3 \\ x_5 & x_1 + x_6 & x_2 \end{pmatrix}$$

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But $\mathrm{in}_\prec(I_2(L))$ has a generator in degree 3 for every $\prec$ with $x_1 \succ x_2 \succ \cdots \succ x_6$

$\Longrightarrow$ the 2-minors are not a UGB

Our generalizations

Matrices of linear forms that are either column or row graded

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$\deg x_{ij} = e_j \in \mathbb{Z}^n$.

$L = (L_{ij})$ with $\deg L_{ij} = e_j$, that is, $L_{ij} = \sum_{k=1}^{m} \lambda_{ijk} x_{kj}, \ \lambda_{ijk} \in K$.

Example: $$L = \begin{pmatrix} x_{11} & 0 & x_{13} - 2x_{23} & -x_{24} \\ 0 & x_{12} + x_{22} & x_{23} & -x_{24} \end{pmatrix}$$

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Example: $$L = \begin{pmatrix} x_{11} & x_{11} + x_{12} & x_{11} - x_{12} & x_{14} \\ 0 & x_{21} & x_{21} + 4x_{24} & x_{24} \end{pmatrix}$$

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Remarks :

- the generators of $I_m(L)$ have all distinct multidegrees
- If $\mathrm{height}(I_m(L)) = n - m + 1$, Theorem 1 can be proved with arguments similar to the ones used for $I_m(X)$

Row-graded

Main difference with the column-graded case:
all the minors are in the same multidegree $\longrightarrow$ we cannot expect that the maximal minors are a universal GB since they might have all the same initial term.

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Example

Consider $K[x_1, \ldots, x_6]$ multigraded by
$\deg(x_1) = \deg(x_3) = \deg(x_5) = (1, 0)$
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$$L = \begin{pmatrix} x_1 & 2x_1 + x_3 & -x_1 + x_5 \\ x_2 & x_2 + x_4 & x_2 + x_6 \end{pmatrix}$$

The 2 minors of L have all degree $(1, 1)$.

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If $x_1 \succ x_2 \succ \ldots \succ x_6$, then $\mathrm{in}_{\prec}(f) = x_1 x_2$ for every 2-minor f
Thus the minors cannot be a universal GB!

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Remarks :

- Experiments show that perhaps the assumption on the height is superfluous
- Main tool of the proof of Theorems 1 and 2: a rigidity property of multigraded generic initial ideals

Generic initial ideal

Theorem/Definition

$\mathrm{GL}_n(K)$ acts by linear substitution on $R = K[x_1, \ldots, x_n]$.
For $g \in \mathrm{GL}_n(K)$ and $I \subset R$ consider $g(I)$

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Properties:

- $\mathrm{gin}(I)$ is Borel fixed, that is, fixed by the upper triangular matrices in $\mathrm{GL}_n(K)$
- $\mathrm{HS}(I, y) = \mathrm{HS}(\mathrm{gin}(I), y)$

Multigraded generic initial ideal

$R = K[x_{ij} : i = 1, \ldots, m \text{ and } j = 1 \ldots, n_i]$ multi graded by $\deg(x_{ij}) = e_i \in \mathbb{Z}^m$ for all $j = 1, \ldots, n_i$.

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(Multigraded) Hilbert series of a multi graded R-module M:

$$\mathrm{HS}(M, y) = \mathrm{HS}(M, y_1, \ldots, y_m) = \sum_{a \in \mathbb{Z}^m} (\dim M_a) y^a$$

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Definition

$\mathrm{G} = \mathrm{GL}_{n_1}(K) \times \cdots \times \mathrm{GL}_{n_m}(K)$ acts by linear substitution on R preserving the multigraded structure.
$g \in \mathrm{G}$, $I \subset R$ multigraded ideal $\rightarrow g(I)$ (multigraded)

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$$U = \{(b_1, \ldots, b_m) \in \mathbb{N}^m : b_i \leq n_i \text{ for every } i = 1, \ldots, m\}.$$

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Idea of the proof:

- I Borel fixed $\Rightarrow$ $\mathrm{Min}(I)$ are Borel fixed
- explicit description of prime Borel fixed ideals $\Rightarrow$ $\mathrm{HS}(I, y)$ determines $\mathrm{Min}(I)$
- $\mathrm{Min}(I) = \mathrm{Min}(J)$, I radical $\Rightarrow$ $J \subseteq I$
- $=$ is forced by $\mathrm{HS}(I, y) = \mathrm{HS}(J, y)$

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To prove Theorems 1-2:
have a guess for $\mathrm{multigin}(I_m(L)) \to$ call it I.
apply corollary with $J = I_m(L)$ and with $J = \mathrm{in}(I_m(L))$

Proof of Theorem 1 - 2

Fix any term order with (wlg):
$x_{1j} \succ x_{ij}$ if $i > 1$ (Thm 1) or $x_{ij} \succ x_{ik}$ if $k > j$ (Thm 2)

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- it is radical and Borel fixed (easy)
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(Hard and different in case 1 and 2)
Combinatorial tools: manipulation of power series expansions involving symmetric polynomials

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$\Rightarrow$ "$=$" holds.
Projective dimension: I is the Alexander dual of a complex related to a matroid, whose regularity is known

Thanks for the attention!