Algebraic properties and invariants of lattice and matrix ideals

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Notation

- $S = K[t_1, \ldots, t_s]$ polynomial ring over a field K
- I an ideal of S and S/I its quotient ring.
- The K-vector space of polynomials in S (resp. I) of degree at most i is denoted by $S_{\leq i}$ (resp. $I_{\leq i}$).

- The *affine Hilbert function* of S/I is:

$$H_I^a(i) := \dim_K(S_{\leq i}/I_{\leq i}), \quad i = 0, 1, 2, \ldots,$$

- (Hilbert-Serre Theorem) There is a unique polynomial $h_I^a(t) \in \mathbb{Q}[t]$ of degree $k \geq 0$ such that

$$h_I^a(i) = H_I^a(i) \text{ for } i \gg 0.$$

- The positive integer

$$\deg(S/I) := k! \lim_{i \to \infty} H_I^a(i)/i^k$$

is the *degree* of S/I and k is the dimension of S/I.

Definition

- If $a = (a_1, \ldots, a_s) \in \mathbb{N}^s$, we set $t^a := t_1^{a_1} \cdots t_s^{a_s}$.
- A *binomial ideal* is an ideal $I \subset S$ generated by "binomials", i.e., by polynomials of the form $t^a - t^b$.
- If $a \in \mathbb{Z}^s$, we can write $a = a^+ - a^-$, with a^+, a^- in $\mathbb{N}^s$.
- A *lattice ideal* is an ideal of the form

$$I(\mathcal{L}) := (t^{a^+} - t^{a^-} \mid a \in \mathcal{L}) \subset S,$$

  where $\mathcal{L}$ is a lattice (subgroup) of $\mathbb{Z}^s$.
- If $P = I(\mathcal{L}) \subset S$, where $\mathcal{L} = \ker_{\mathbb{Z}}(A)$ for some integer matrix A, then P is called a *toric ideal*.

Let $\mathcal{L} \subset \mathbb{Z}^s$ be a lattice.

- The *torsion subgroup* of $\mathbb{Z}^s/\mathcal{L}$ is:

$$T(\mathbb{Z}^s/\mathcal{L}) := \{\overline{x} \in \mathbb{Z}^s/\mathcal{L} \mid \ell\,\overline{x} = \overline{0} \text{ for some } \ell \in \mathbb{N}_+\}.$$

- The following hold:

  (a) $\mathrm{ht}(I(\mathcal{L})) = \mathrm{rank}(\mathcal{L})$,

  (b) $I(\mathcal{L})$ is unmixed,

  (c) $\mathbb{Z}^s/\mathcal{L} \simeq \mathbb{Z}^{s-r} \oplus T(\mathbb{Z}^s/\mathcal{L}),\ \ r = \mathrm{rank}(\mathcal{L})$.

Theorem (O'Carroll, Planas-Vilanova, -, 2014)

Let $\mathcal{L} \subset \mathbb{Z}^s$ be a lattice of rank r. The following hold.

(a) If $r < s$, there is an integer matrix A of size $(s-r) \times s$ and rank $s-r$ such that $\mathcal{L} \subset \ker_{\mathbb{Z}}(A)$.

(b) If $r < s$ and $v_1, \ldots, v_s$ are the columns of A, then

$$\deg(S/I(\mathcal{L})) = \frac{|T(\mathbb{Z}^s/\mathcal{L})|(s-r)!\mathrm{vol}(\mathrm{conv}(0, v_1, \ldots, v_s))}{|T(\mathbb{Z}^{s-r}/\mathbb{Z}\{v_1, \ldots, v_s\}|}.$$

(c) If $r = s$, then $\deg(S/I(\mathcal{L})) = |\mathbb{Z}^s/\mathcal{L}|$.

- Assume $\mathbf{d} = (d_1, \ldots, d_s) \in \mathbb{N}_+^s$. Then $S = \oplus_{d=0}^{\infty} S_d$ has a grading induced by setting $\deg(t_i) = d_i$.
- A graded ideal of S is an ideal which is graded with respect to $\mathbf{d}$.
- If $d_i = 1$ for all i and $I \subset S$ is graded, then

$$H_I(i) := H_I^a(i) - H_I^a(i-1) = \dim_K(S_i/I_i).$$

Corollary (López, -, 2013)

If $\dim(S/I(\mathcal{L})) = 1$ *and* $I(\mathcal{L})$ *is graded, then*

$$(\gcd\{d_i\}_{i=1}^{s}) \deg(S/I(\mathcal{L})) = (\max\{d_i\}_{i=1}^{s}) |T(\mathbb{Z}^s/\mathcal{L})|.$$

Example

If $\mathcal{L} = \ker_{\mathbb{Z}}(1, 2, 3)$, then

$$\deg(S/I(\mathcal{L})) = 3.$$

Lemma

Let I be a graded binomial ideal and let $V(I, t_i)$ be the *affine variety* *defined by the ideal (I, t_i).*

If $V(I, t_i) = \{0\}$ for $i = 1, \ldots, s$, then $\dim(S/I) = 1$.

Example

If I is the ideal of $S = K[t_1, t_2, t_3, t_4]$ generated by

$$\begin{aligned} f_1 &= t_1^3 - t_2 t_3 t_4, \\ f_2 &= t_2^3 - t_1 t_3 t_4, \\ f_3 &= t_3^3 - t_1 t_2 t_4, \\ f_4 &= t_4^3 - t_1 t_2 t_3, \end{aligned}$$

then $V(I, t_i) = \{0\}$ for all i. Hence $\dim(S/I) = 1$.

Theorem (O'Carroll, Planas-Vilanova, -, 2014)

Let $I \subset S$ be a graded binomial ideal. Then I is a lattice ideal of dimension 1 if and only if

(a) $V(I, t_i) = \{0\}$ for $i = 1, \ldots, s$, and

(b) I is an unmixed ideal.

Condition (b) can be replaced by

(b_1) I is Cohen-Macaulay, or by

(b_2) $\mathrm{Hull}(I) = I$.

$\mathrm{Hull}(I) =$ intersection of the isolated primary components.

Definition

A *matrix ideal* is an ideal of the form

$$I(L) = (t^{a_i^+} - t^{a_i^-} \mid i = 1, \dots, m) \subset S,$$

where the a_i's are the columns of an integer matrix L of size $s \times m$. Note that $I(L)$ is graded if and only if $\mathbf{d}L = 0$.

Example

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$I(L) = (t_1^2 - t_2 t_3,\ t_2 - t_3,\ t_3 - t_2), \quad V(I(L), t_i) = \{0\}\ \forall\, i,$$

$$I(L^\top) = (t_1^2 - 1,\ t_2 - t_1 t_3,\ t_3 - t_1 t_2), \quad 0 \notin V(I(L^\top), t_1).$$

Definition

A graded ideal $I \subset S$ is a *complete intersection* if I is generated by $\mathrm{ht}(I)$ homogeneous polynomials.

Problem

Characterize in algebraic and geometric terms the complete intersection matrix ideals of dimension 1.

Theorem (López, -, 2013)

A graded lattice ideal $I(\mathcal{L}) \subset S$ of dimension 1 is a complete intersection if and only if there is an integer matrix L of size $s \times (s-1)$ with column vectors $a_1, \ldots, a_{s-1}$ such that

(a) $V(I(L), t_i) = \{0\}$ for all i, and

(b) $\mathcal{L} = \mathbb{Z}\{a_1, \ldots, a_{s-1}\}$.

Proof. $\Rightarrow$) If $I(\mathcal{L}) = (\{t^{a_i^+} - t^{a_i^-}\}_{i=1}^{s-1})$, then (b) holds. If L is the matrix with column vectors $a_1, \ldots, a_{s-1}$, then (a) holds.

$\Leftarrow$) By (a), $\mathrm{ht}(I(L)) = s - 1$. Hence $I(L)$ is a c.i. and by an earlier result $I(L)$ is a lattice ideal. We claim that $I(L) = I(\mathcal{L})$. The inclusion "$\subset$" follows from (b). There is t^δ such that $t^\delta I(\mathcal{L}) \subset I(L)$. Thus $I(\mathcal{L}) \subset I(L)$. $\square$

The next corollary follows using a result of Katsabekis, Morales and Thoma [J. Algebra **324** (2010)].

Corollary

If $\mathrm{char}(K) = p > 0$ *and* $I \subset S$ *is a graded lattice ideal of dimension* 1*, then there are binomials* $g_1, \ldots, g_r$ *such that*

$$\mathrm{rad}(I) = \mathrm{rad}(g_1, \ldots, g_r),$$

where r *is the height of* I*.*

Graded matrix ideals satisfying $V(I, t_i) = \{0\}$

Proposition (O'Carroll, Planas-Vilanova, -, 2014)

Let $I = I(L)$ be a graded matrix ideal and $\mathcal{L}$ the lattice spanned by the columns of L. Suppose $V(I, t_i) = \{0\}\ \forall\ i$.

(a) *If I is unmixed, then $I = I(\mathcal{L})$.*

(b) *If I is not unmixed, then $I = I(\mathcal{L}) \cap \mathfrak{q}$, where $\mathfrak{q}$ is an $\mathfrak{m}$-primary component of I.*

(c) $(\gcd\{d_i\}_{i=1}^{s}) \deg(S/I) = \max_i\{d_i\} |T(\mathbb{Z}^s/\mathcal{L})|$.

Definition

Let $a_{i,j} \in \mathbb{N}$, $i, j = 1, \ldots, s$, and let L be an $s \times s$ matrix of the following special form:

$$L = \begin{pmatrix} a_{1,1} & -a_{1,2} & \cdots & -a_{1,s} \\ -a_{2,1} & a_{2,2} & \cdots & -a_{2,s} \\ \vdots & \vdots & \ldots & \vdots \\ -a_{s,1} & -a_{s,2} & \cdots & a_{s,s} \end{pmatrix}.$$

The matrix L is called a *pure binomial* matrix (*PB* matrix) if $a_{j,j} > 0$ for all j, and for each row and each column of L at least one off-diagonal entry is non-zero.

The Laplacian matrix of a graph is a PB matrix.

Next, we study PB matrices using algebraic graph theory.

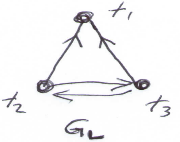
Definition

Let $B = (b_{i,j})$ be an $s \times s$ real matrix. The *underlying digraph* of B, denoted by G_B, has vertex set $\{t_1, \ldots, t_s\}$, with an arc from t_i to t_j if and only if $b_{i,j} \neq 0$. If $b_{i,i} \neq 0$, we put a loop at vertex t_i. If B is a PB matrix, loops in G_B are omitted.

A digraph is *strongly connected* if for any two vertices t_i and t_j there is a directed path form t_i to t_j and a direct path from t_j to t_i.

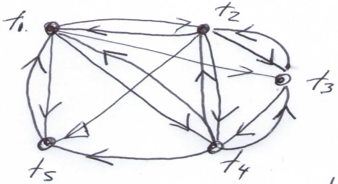
Examples of underlying digraphs

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$



underlying digraph not strongly connected.

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



$$L \cdot (1,1,1,1,1)^T = 0$$

$$(4,3,5,3,10)L = 0$$

G_L is strongly connected.

Duality Theorem (O'Carroll, Planas-Vilanova, -, 2014)

Let L be a PB matrix of size $s \times s$ such that $\mathbf{d}L = 0$ for some $\mathbf{d} \in \mathbb{N}_+^s$.

(a) If G_L is strongly connected, then $\mathrm{rank}(L) = s - 1$ and there is $\mathbf{c}$ in $\mathbb{N}_+^s$ such that $L\mathbf{c}^\top = 0$.

(b) The following conditions are equivalent:

(b_1) G_L is strongly connected.

(b_2) $V(I(L), t_i) = \{0\}$ for all i.

(b_3) $L_{i,j} > 0 \ \forall\, i, j$; $\mathrm{adj}(L) = (L_{i,j})$ is the adjoint of L.

Example

Let L be the following PB matrix and $\mathrm{adj}(L)$ its adjoint:

$$L = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{adj}(L) = \begin{pmatrix} 20 & 15 & 25 & 15 & 50 \\ 20 & 15 & 25 & 15 & 50 \\ 20 & 15 & 25 & 15 & 50 \\ 20 & 15 & 25 & 15 & 50 \\ 20 & 15 & 25 & 15 & 50 \end{pmatrix}.$$

By last theorem, we get $V(I(L), t_i) = \{0\}$ for all i and G_L is strongly connected.

Complementing a result of Herzog

Theorem (O'Carroll, Planas-Vilanova, -, 2014)

If $S = K[t_1, t_2, t_3]$, then I is a graded lattice ideal of S of dimension 1 if and only if I is the matrix ideal of a 3×3 PB matrix L such that $L\mathbf{1}^\top = 0$.

This result is due to Herzog [Manuscripta Math., 1970] when I is the toric ideal

$$I = (t^{a^+} - t^{a^-} \mid a \in \ker_{\mathbb{Z}}(d_1, d_2, d_3))$$

of the monomial space curve parameterized by $y_1^{d_1}, y_1^{d_2}, y_1^{d_3}$, where d_1, d_2, d_3 are positive integers.

Example

Let L be the PB matrix

$$L = \begin{pmatrix} 4 & -2 & -2 \\ -1 & 4 & -3 \\ -2 & -2 & 4 \end{pmatrix}.$$

Since $L\mathbf{1}^\top = 0$, by the last theorem we get that

$$I(L) = (t_1^4 - t_2 t_3^2, t_2^4 - t_1^2 t_3^2, t_1^2 t_2^3 - t_3^4)$$

is a graded lattice ideal of dimension 1 which is graded with respect to $\mathbf{d} = (5, 6, 7)$.

The degree of $S/I(L)$ is 14. If $K = \mathbb{Q}$, then $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$, where $\mathfrak{p}_1, \mathfrak{p}_2$ are prime ideals of degree 7.

Laplacian matrices and ideals

Let G be a connected simple graph, where $V = \{t_1, \dots, t_s\}$ is the set of vertices, E is the set of edges, and let w be a weight function

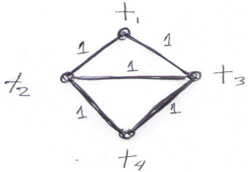
$$w\colon E \to \mathbb{N}_+, \quad e \mapsto w_e.$$

Let $E(t_i)$ be the set of edges incident to t_i. The *Laplacian matrix* $L(G)$ of G is the $s \times s$ matrix whose (i, j)-entry $L(G)_{i,j}$ is given by

$$L(G)_{i,j} := \begin{cases} \sum_{e \in E(t_i)} w_e & \text{if } i = j, \\ -w_e & \text{if } i \neq j \text{ and } e = \{t_i, t_j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

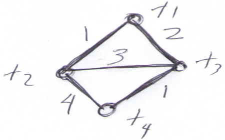
Notice that $L(G)$ is symmetric, $\mathbf{1}L(G) = 0$ and $\operatorname{rank}(L(G)) = s - 1$.

Weighted graphs as multigraphs



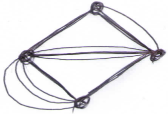
$$L(G) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

(i,j) entry of Adjoint $= 8$



$$L(G) = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 8 & -3 & -4 \\ -2 & -3 & 6 & -1 \\ 0 & -4 & -1 & 5 \end{bmatrix}$$

(i,j) entry of Adjoint $= 67$



We can regard the weighted graph G as a multigraph (each edge e occurs w_e times)

Definition

- The matrix ideal I_G of $L(G)$ is called the *Laplacian ideal* of the graph G.
- If $\mathcal{L}$ is the lattice generated by the columns of $L(G)$, the group
$$K(G) := T(\mathbb{Z}^s/\mathcal{L})$$
is called the *critical group* or the *sandpile group* of G.

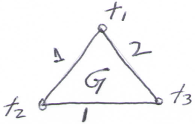
- If G is connected, then $\mathrm{rank}(\mathcal{L}) = \mathrm{ht}(I_G) = s - 1$.
- The structure, as a finite abelian group, of $K(G)$ is only known for a few families of graphs.

Theorem (the Matrix-Tree Theorem, Cayley (1889))

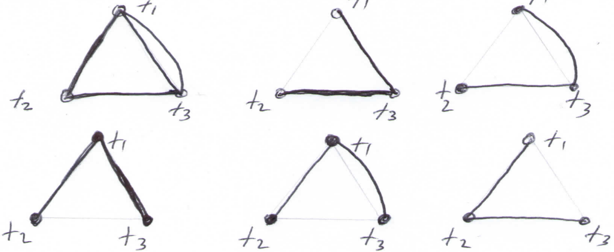
If G is regarded as a multigraph (where each edge e occurs w_e times), then the number of spanning trees of G is the (i,j)-entry of the adjoint matrix of $L(G)$ for any (i,j).

The order of $T(\mathbb{Z}^s/\mathcal{L})$ is the gcd of all $(s-1)$-minors of $L(G)$. Hence the order of $K(G) = T(\mathbb{Z}^s/\mathcal{L})$ is the number of spanning trees of G.

THE MATRIX-TREE THEOREM



$$L(G) = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & -3 \end{bmatrix}$$



G has 5 spanning trees

Proposition

Let G be a connected graph, $I_G \subset S$ its Laplacian ideal, $\mathcal{L}$ the lattice spanned by the columns of $L(G)$. Then:

(a) $V(I_G, t_i) = \{0\}$ *for all* i.

(b) $\deg(S/I_G) = \deg(S/I(\mathcal{L})) = |K(G)|$.

(c) $\mathrm{Hull}(I_G) = I(\mathcal{L})$.

(d) *If* $\deg_G(t_i) \geq 3$ *for all* i*, then* I_G *is not a lattice ideal.*

(e) *If* $\deg_G(t_i) \geq 2$ *for all* i*, then* I_G *is not a complete intersection.*

(f) *If* $G = \mathcal{K}_s$ *is a complete graph, then* $\deg(S/I_G) = s^{s-2}$.

Example

Let G be the complete graph on 3 vertices. Then

$$L(G) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$I_G = (t_1^2 - t_2 t_3,\ t_2^2 - t_1 t_3,\ t_3^2 - t_1 t_2), \quad V(I_G, t_i) = \{0\}\ \forall\, i.$$

I_G is not a complete intersection, is a lattice ideal, is not a prime ideal, has degree 3 and dimension 1.

Vanishing ideals over finite fields

Let $K = \mathbb{F}_q$ be a finite field with q elements and X a subset of a projective space $\mathbb{P}^{s-1}$ over K.

The following family arises in algebraic coding theory.

Proposition (Vaz Pinto, Neves, -, 2013)

The vanishing ideal $I(X)$ is a lattice ideal of S if and only if

$$X = \{[(x^{v_1}, \ldots, x^{v_s})] | \, x_i \in K^* = K \setminus \{0\} \; \forall \, i\} \subset \mathbb{P}^{s-1},$$

for some monomials $x^{v_i} := x_1^{v_{i1}} \cdots x_n^{v_{in}}, \quad i = 1, \ldots, s.$

In what follows we assume that X is a subset of $\mathbb{P}^{s-1}$ *parameterized by monomials*:

$$X = \{[(x^{v_1}, \ldots, x^{v_s})] \,|\, x_i \in K^* = K \setminus \{0\} \; \forall\, i\} \subset \mathbb{P}^{s-1}.$$

Some properties of $I(X)$

(b) $I(X) = (\{t_i - x^{v_i} z\}_{i=1}^{s} \cup \{x_i^{q-1} - 1\}_{i=1}^{n}) \cap S.$

(c) $I(X) = I(\mathcal{L})$ for some lattice $\mathcal{L}$ of $\mathbb{Z}^s$ of rank $s-1$.

(d) $\deg(S/I(X) = |T(\mathbb{Z}^s/\mathcal{L})|.$

(e) If $x^{v_1}, \ldots, x^{v_s}$ are square-free, then $I(X)$ is a complete intersection if and only if X is a projective torus.

It is usual to denote $H_{I(X)}$ simply by H_X.

- The degree of $S/I(X)$ is $|X|$.
- The least integer $r \geq 0$ such that $H_X(d) = |X|$ for $d \geq r$ is called the *regularity* of $S/I(X)$ and is denoted by $\mathrm{reg}(S/I(X))$.

Parameterized linear codes arising from X

Definition

Let $X = \{[P_1], \ldots, [P_m]\}$ and let $f_0(t_1 \ldots, t_s) = t_1^d$, where $d \geq 1$. The linear map of K-vector spaces:

$$\mathrm{ev}_d\colon S_d \to K^{|X|}, \qquad f \mapsto \left(\frac{f(P_1)}{f_0(P_1)}, \ldots, \frac{f(P_m)}{f_0(P_m)}\right)$$

is called an *evaluation map*. Notice that $\ker(\mathrm{ev}_d) = I(X)_d$.

- The image of ev_d, denoted by $C_X(d)$, is a *linear code*
- $C_X(d)$ is called a *parameterized code* of order d.

Definition

The basic parameters of the linear code $C_X(d)$ are:

- $\dim_K C_X(d) = H_X(d)$, the *dimension*,
- $|X| = \deg(S/I(X))$, the *length*,
- $\delta_X(d) := \min\{\|v\| : 0 \neq v \in C_X(d)\}$, the *minimum distance*, where $\|v\|$ is the number of non-zero entries of v.

$$\delta_X(d) = 1 \text{ for } d \geq \operatorname{reg}(S/I(X)).$$

Main Problem:

Find formulas, in terms of n, s, q, d, and the "combinatorics" of $x^{v_1}, \ldots, x^{v_s}$, for the *basic parameters*:

(a) $H_X(d)$,

(b) $\deg(S/I(X))$,

(c) $\delta_X(d)$,

(d) $\operatorname{reg}(S/I(X))$.

Formulas for $(a) - (d)$ are known when:

- (Sarmiento, Vaz Pinto, -, 2011) X is a projective torus.
- (López, Rentería, -, 2014) X is a set parameterized by $x_1^{d_1}, \ldots, x_s^{d_s}, 1$, where $d_i \in \mathbb{N}_+$ for al i

Example

If $X = \{[(x_1^{90}, x_2^{36}, x_3^{20}, 1)] \mid x_i \in \mathbb{F}_{181}^* \text{ for } i = 1, 2, 3\}$, then

$\operatorname{reg}(K[t_1, \ldots, t_4]/I(X)) = 13$.

The basic parameters of the family $\{C_X(d)\}_{d \geq 1}$ are:

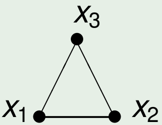
d	1	2	3	4	5	6	7	8	9	10	11	12	13
$\|X\|$	90	90	90	90	90	90	90	90	90	90	90	90	90
$H_X(d)$	4	9	16	25	35	45	55	65	74	81	86	89	90
$\delta_X(d)$	45	36	27	18	9	8	7	6	5	4	3	2	1

Definition (Parameterizations arising from graphs)

Let G be a graph with vertices $x_1, \ldots, x_n$. The set parameterized by G is the set X parameterized by all $x_i x_j$ such that $\{x_i, x_j\}$ is an edge of G.

Example

Let G be the graph:



Then $X = \{[(x_1x_2, x_2x_3, x_1x_3)] \mid x_i \in K^* \text{ for all } i\} \subset \mathbb{P}^2$.

Remark

Let G be a connected graph.

- Formulas for the basic parameters of $C_X(d)$ are known for complete bipartite graphs.
- For complete graphs, no formula is known for the minimum distance.

Next we show a formula for the degree for any graph G.

Theorem (Neves, Vaz Pinto,-, 2012)

Let G be a graph with n vertices, c connected components and γ non-bipartite components. Then

$$\deg(S/I(X)) = \begin{cases} \left(\frac{1}{2}\right)^{\gamma-1}(q-1)^{n-c+\gamma-1}, & \text{if } \gamma \geq 1,\ q \text{ odd}, \\ (q-1)^{n-c+\gamma-1}, & \text{if } \gamma \geq 1,\ q \text{ even}, \\ (q-1)^{n-c-1}, & \text{if } \gamma = 0. \end{cases}$$

The end