Minimal Triangulations & Graded Betti Numbers

OSAKA UNIVERSITY
Live Locally, Grow Globally

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Meeting On Combinatorial Commutative Algebra 2014

summary of slide is available at my homepage

Minimal Triangulations

Question

Let M be a topological manifold. How many vertices do we need to triangulate M? In particular, what is the smallest number of the vertices?

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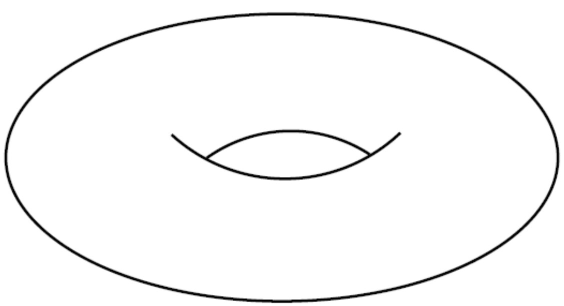
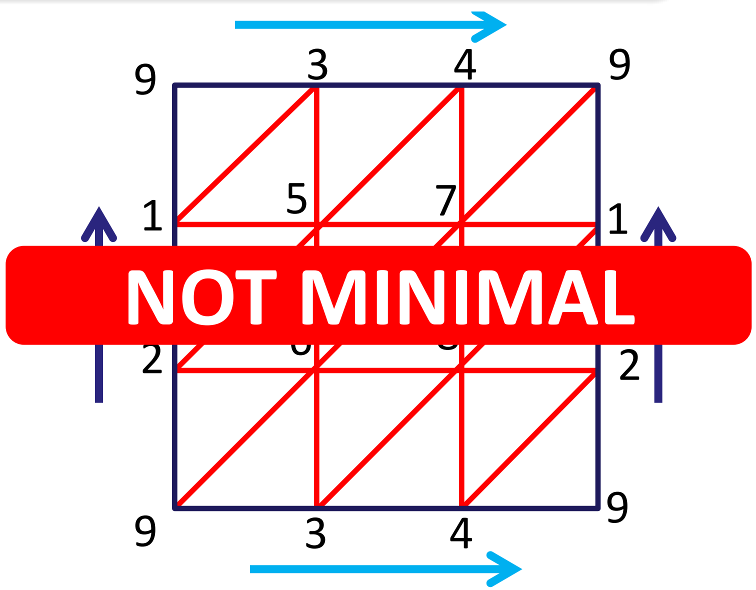


Torus $S^1 \times S^1$



9 vertex triangulation

Minimal Triangulations

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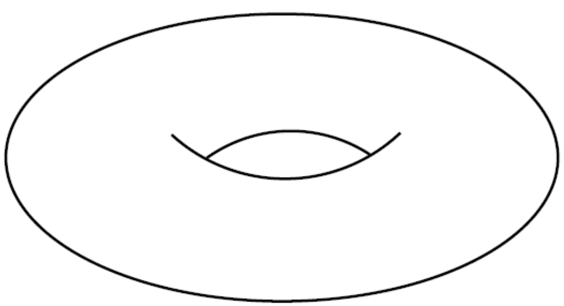

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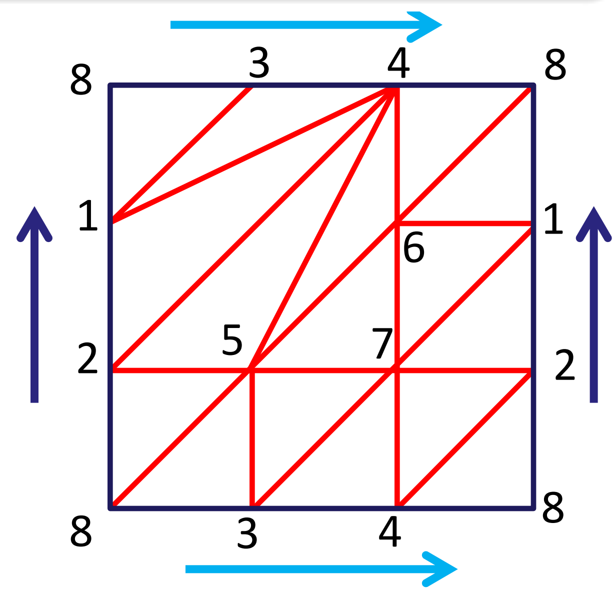

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Minimal Triangulations

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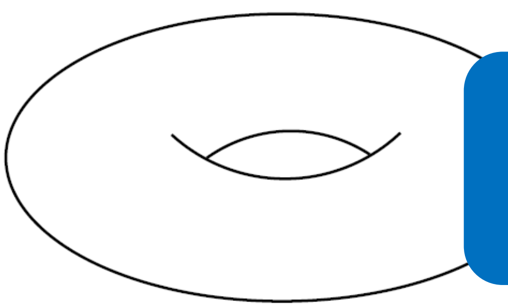
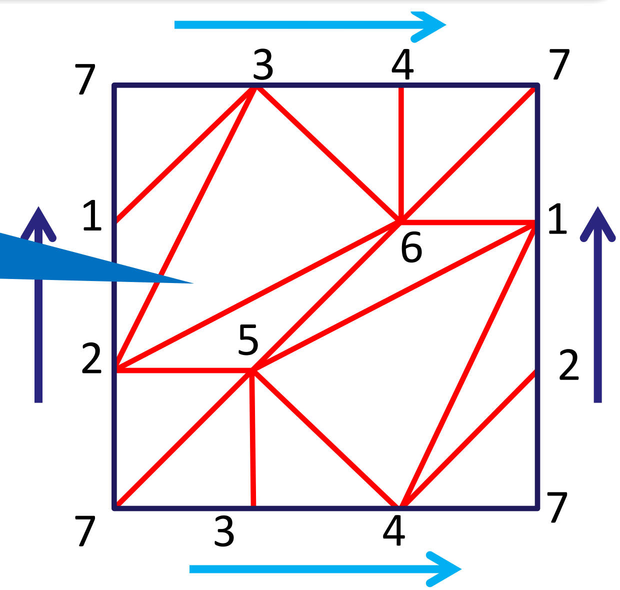


Császár Torus

Torus $S^1 \times S^1$



7 vertex triangulation

Minimal triangulations of closed surfaces

Closed surfaces (2-manifolds)

The topological type of a connected closed surface are completely classified. They are either

(1)

(2)

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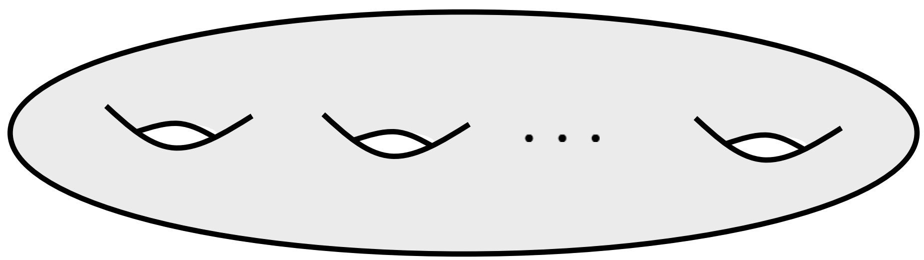

(2)

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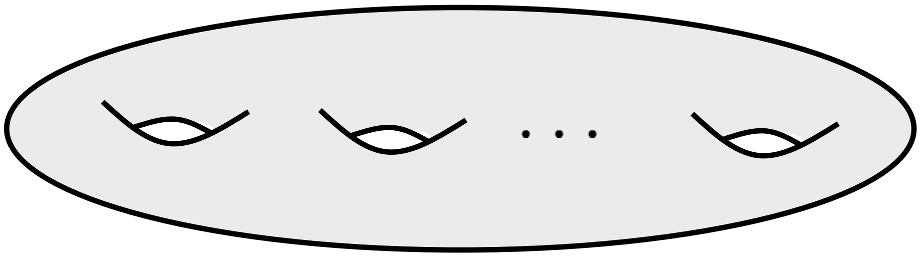
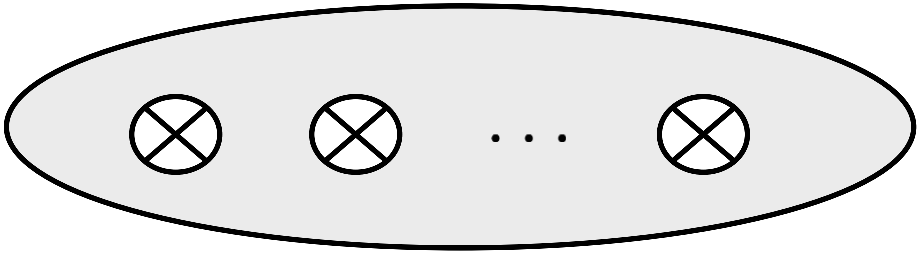


(2) $N_g = (\mathbb{R}P^2)\# \cdots \#(\mathbb{R}P^2)$



Minimal triangulations of surfaces

M: connected closed surface

Theorem (Heawood 1890)

If an n vertex triangulation of M exists, then

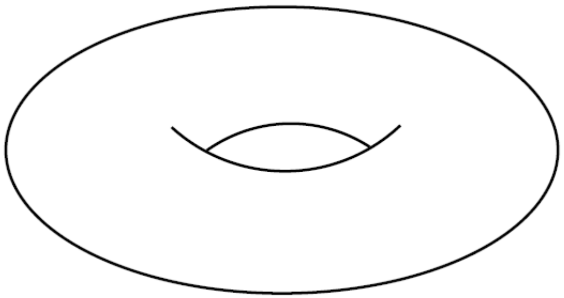
$$\binom{n-3}{2} \geq 3 \cdot (2 - \chi(M)). \tag{1}$$

Theorem (Ringel '55, Jungerman–Ringel '80)

If $M \neq S_2, N_2$ or N_3 then an n vertex triangulation of M exists if and only if (1) holds.

Example 1

$$S_1 = S^1 \times S^1 :$$



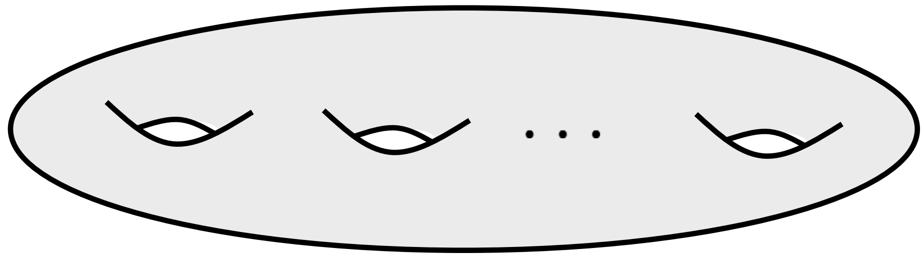
For S_1, Heawood's inequality is

$$\binom{n-3}{2} \geq 3 \times 2 = 6$$

Minimal triangulation has 7 vertices. $\left(\binom{4}{2} = 6\right)$

Example 2

$S_{100} = (S^1 \times S^1)^{\#100}$:



For S^{100} , Heawood's inequality is

$$\binom{n-3}{2} \geq 3 \times 200 = 600$$

Minimal triangulation has 39 vertices.

$$\left(\binom{35}{2} = 595,\ \binom{36}{2} = 630\right)$$

Minimal triangulations of closed surfaces

M: connected closed surface

Theorem (Heawood 1890)

If an n vertex triangulation of M exists, then

$$(1) \qquad \binom{n-3}{2} \geq 3 \cdot (2 - \chi(M)).$$

Theorem (Ringel '55, Jungerman–Ringel '80)

If $M \neq S_2, N_2$ or N_3 then an n vertex triangulation of M exists if and only if (1) holds.

Lower bounds of the number of the vertices & Upper bounds of graded Betti numbers

Notation 1

- Δ : (abstract) simplicial complex with n vertices
- M : connected closed d-mfd
- $|\Delta|$: geometric realization of Δ
- Δ is a triangulation of $M \Leftrightarrow |\Delta| \cong_{\mathrm{homeo}} M$
- M is orientable $\Leftrightarrow H_d(M;\mathbb{Z}) \cong \mathbb{Z}$

Notation 2

- $\mathrm{lk}_\Delta(v) = \{F \in \Delta : v \notin F,\ \{v\} \cup F \in \Delta\}$
- Δ is a **combinatorial triangulation** of M
  $\Leftrightarrow$ $|\Delta| \cong_{\mathrm{homeo}} M$ and each $\mathrm{lk}_\Delta(v)$ is a PL sphere.
- Δ is a **$\mathbb{F}$-homology manifold**
  $\Leftrightarrow$ pure & each $\mathrm{lk}_\Delta(v)$ is Gorenstein* (over $\mathbb{F}$)

Combinatorial Triangulations $\subset$ Triangulations of closed mfds $\subset$ Homology manifolds

Generalizations of Heawood's inequality

Conjecture (Kühnel)

If an n vertex (combinatorial) triangulation of a connected closed d-mfd M exists, then

$$\binom{n-d+j-2}{j+1} \geq \binom{d+2}{j+1} \times (\dim_{\mathbb{F}} H_j(M;\mathbb{F}))$$

for $j < \frac{d}{2}$, and (when d is even)

$$\binom{n-\frac{d}{2}-2}{\frac{d}{2}+1} \geq \binom{d+2}{\frac{d}{2}+1} \times \frac{1}{2}(\dim_{\mathbb{F}} H_{\frac{d}{2}}(M;\mathbb{F})).$$

Remark: Heawood's inequality is the special case when $d=2,\ j=1, \mathbb{F}=\mathbb{Z}/2\mathbb{Z}$.

Known cases

Kühnel's Conjecture holds for

- $d = 4,\ j = 2$ (Kühnel '90).
- $d = 3,\ j = 1$ (Bagchi '14).
- $j = \frac{d}{2}$, M orientable or $\mathrm{char}(\mathbb{F}) = 2$.
- $j = 1$, M orientable or $\mathrm{char}(\mathbb{F}) = 2$.
- M orientable, $\mathrm{char}(\mathbb{F}) = 0$, each $\mathrm{lk}_\Delta(v)$ is a polytope. (Novik–Swartz '09).

Result

Theorem (M)

Kühnel's conjecture holds for the following cases;

- $j = 1$
- $j = \frac{d}{2}$ *(d: even)*
- *triangulations* Δ *such that each* $\mathrm{lk}_{\Delta}(v)$ *is a polytope when* $\mathrm{char}(\mathbb{F}) = 0$.

Idea of Proof

$\mathbb{F}[\Delta] = \mathbb{F}[x_v : v \in \mathrm{Vert}(\Delta)]/I_\Delta$: Stanley–Reisner ring

$I_\Delta = (x_{v_1} x_{v_2} \cdots x_{v_k} : \{v_1, v_2, \ldots, v_k\} \notin \Delta)$

$\beta_{i,j}(I_\Delta) = \dim_{\mathbb{F}} \mathrm{Tor}_i(I_\Delta, \mathbb{F})_j$: graded Betti number

Observation

Let n_v be the num of vertices of $\mathrm{lk}_\Delta(v)$. If

$$\beta_{i,i+j+1}(I_{\mathrm{lk}_\Delta(v)}) \leq \beta_{i,i+j+1}\left((x_1, \ldots, x_{n_v-d-1})^{j+1}\right)$$

for any i and for any vertex v, then

$$\binom{n-d-1}{2} \geq \binom{d+2}{2} \times \left(\dim_{\mathbb{F}} H_j(M;\mathbb{F})\right)$$

Question

Question

Let Δ be a Gorenstein simplicial complex of dimension $d-1$ with n vertices. Does*

$$\beta_{i,i+j}(I_\Delta) \le \beta_{i,i+j}\big((x_1,\dots,x_{n-d-1})^j\big)$$

holds for all i and $j \le \frac{d+1}{2}$?

- The bounds hold for $j = 2$.
- The bounds hold for simplicial polytopes ($\mathrm{char}(\mathbb{F}) = 0$). (essentially Migliore–Nagle '03)

Tight triangulations
&
Linear resolutions

Definition

$\Delta_W = \{F \in \Delta : F \subset W\}$: induced subcomplex

Definition

A simplicial complex Δ on V is **$\mathbb{F}$-tight** *if a natural map induced from the inclusion*

$$\imath : H_i(\Delta_W; \mathbb{F}) \to H_i(\Delta; \mathbb{F})$$

is injective for all i and $W \subset V$.

We say that Δ is tight if it is $\mathbb{F}$-tight for some field $\mathbb{F}$.

Properties

$$\boxed{\mathbb{F}\text{-tight} \Rightarrow \imath : H_0(\Delta_W;\mathbb{F}) \to H_0(\Delta;\mathbb{F}) \text{ injective } {}^\forall i,\ {}^\forall W}$$

- connected & tight $\Rightarrow$ neighborly

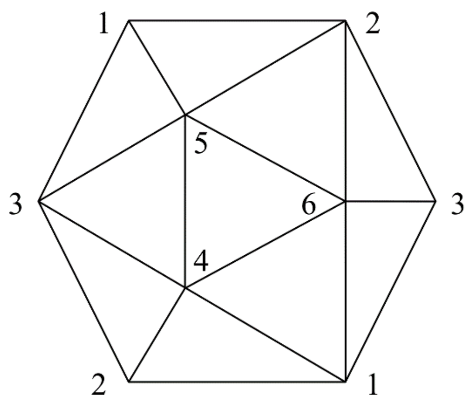
  $({}^\forall u, v\colon \text{ vertices} \Rightarrow \{u, v\} \in \Delta)$

- If Δ is a triangulation of a closed surface, then

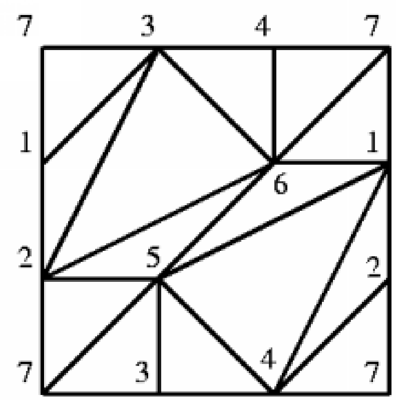
  $$\Delta \text{ is tight} \Leftrightarrow \Delta \text{ is neighborly}$$

Properties

Tight triangulation of $\mathbb{R}P^2$



Tight triangulation of $S^1 \times S^1$



- If Δ is a triangulation of a closed surface, then

$$\Delta \text{ is tight} \Leftrightarrow \Delta \text{ is neighborly}$$

Big problem

Conjecture (Kühnel–Lutz '99)

A tight combinatorial triangulation of a closed manifold M has the smallest number of vertices among all combinatorial triangulations of M.

Motivation of the conjecture

Theorem

An n vertex triangulation of a closed surface (2-mfd) M satisfies

$$\binom{n-3}{2} = 3(2-\chi(M))$$

if and only if it is ~~neighborly~~. **tight**

- Minimal triangulation of $S_1 = S^1 \times S^1$ is tight.
- Minimal triangulation of S_{100} is not tight.

Target Problem

Conjecture (Kühnel–Lutz '99)

A combinatorial triangulation of $S^i \times S^j$ $(i \leq j)$ *is tight if and only if it has* $i + 2j + 4$ *vertices.*

Theorem (Brehm–Kühnel '86)

A combinatorial triangulation of $S^i \times S^j$ *has at least* $i + 2j + 4$ *vertices.*

Result

Conjecture (Kühnel–Lutz '99)

A combinatorial triangulation of $S^i \times S^j$ $(i \leq j)$ *is tight if and only if it has* $i + 2j + 4$ *vertices.*

Theorem (M)

Suppose $j > 2i$. *If a combinatorial triangulation of* $S^i \times S^j$ *is tight, then it has exactly* $i + 2j + 4$ *vertices.*

Tightness & Betti numbers

Δ: simplicial complex on V

$$\sigma_k(\Delta;\mathbb{F}) = \sum_{W\subset V} \frac{1}{\binom{\#V}{\#W}} \dim_{\mathbb{F}} \widetilde{H}_{k-1}(\Delta_W;\mathbb{F})$$

$$\mu_k(\Delta;\mathbb{F}) = \sum_{v\in V} \frac{\sigma_{k-1}(\mathrm{lk}_\Delta(v);\mathbb{F})}{(\text{num of vert of } \mathrm{lk}_\Delta(v)) + 1}$$

Theorem (Bagchi–Datta '14, Bagchi '14)

A simplicial complex Δ is $\mathbb{F}$-tight if and only if $\dim_{\mathbb{F}} H_k(\Delta;\mathbb{F}) = \mu_k(\Delta;\mathbb{F})$ for all k.

Tightness & Betti numbers

Δ: simplicial complex on V

$$\sigma_k(\Delta;\mathbb{F}) = \sum_{\ell=k}^{n} \frac{1}{\binom{n}{\ell}} \beta_{\ell-k,(\ell-k)+k}(\mathbb{F}[\Delta])$$

$\boxed{n = \#V}$

Theorem (Hochster's formula)

$$\beta_{i,j}(\mathbb{F}[\Delta]) = \sum_{\#W=j} \dim_{\mathbb{F}} \widetilde{H}_{i-1}(\Delta_W;\mathbb{F}).$$

Tightness & Betti numbers

Δ: simplicial complex on V

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A simplicial complex Δ is $\mathbb{F}$-tight if and only if $\dim_{\mathbb{F}} H_k(\Delta;\mathbb{F}) = \mu_k(\Delta;\mathbb{F})$ for all k.

Observation

$$I_{\langle k\rangle} = (f \in I : \deg f = k)$$

Observation

Let Δ be a tight combinatorial triang. of $S^i \times S^j$. If $(I_{\mathrm{lk}_\Delta(v)})_{\langle i+1\rangle}$ has a linear resolution for each vertex v, then it has exactly $i+2j+4$ vertices.

Lemma

Suppose $j > 2i$. For any tight triang. Δ of $S^i \times S^j$, $(I_{\mathrm{lk}_\Delta(v)})_{\langle i+1\rangle}$ has a linear resolution for each vertex v.

Why link has linear resolution?

Δ: tight triangulation of $S^i \times S^j$ $(j > 2i)$

Betti Diagram of $\mathbb{F}[\mathrm{lk}_\Delta(v)]$

0	1						
i		*	*	...	...	*	
j		*	*	...	...	*	
$i+j$							1

looks linear resolutioin

Why linear resolution determine the number of the vertices?

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Assumption:

- Δ is an combinatorial triangulaiton of $S^i \times S^j$.
- $(I_{\mathrm{lk}_\Delta(v)})_{\langle i+1 \rangle}$ has a linear resolution.

$\Rightarrow (I_{\mathrm{lk}_\Delta(v)})_{\langle i+1 \rangle}$ is Cohen–Macaulay

$\beta_{\ell,k}(\mathbb{F}[\mathrm{lk}_\Delta(v)])$ only depends on i, j, n.

$\Rightarrow$ use $1 = \dim_{\mathbb{F}} H_i(S^i \times S^j; \mathbb{F}) = \mu_i(\Delta; \mathbb{F})$.

Sum of $\beta_{k,k+i}(\mathbb{F}[\mathrm{lk}_\Delta(v)])$.
Only depends on n, i and j.

Questions

Question

Let Δ be a tight triangulation of $S^i \times S^j$ $(i < j)$. Is it true that $(I_{\mathrm{lk}_\Delta(v)})_{\langle i+1 \rangle}$ has a linear resolution?

Question

Let Δ be a tight homology 3-mfd. Does $(I_{\mathrm{lk}_\Delta(v)})_{\langle 2 \rangle}$ has a 2-linear resolution?

Unfortunate Fact

- Existence of tight triangulations are know for

$$S^1 \times S^j \ (j\text{:odd}),\ S^2 \times S^3,\ S^3 \times S^3$$

- Non-Existence of tight triangulations are known for

$$S^1 \times S^j \ (j\text{:even}),\ S^2 \times S^2$$

$$\boxed{\textbf{Please find tight triangulations of } S^2 \times S^j \ (j \geq 5)}$$

Thank you very much for your attention