

Betti Numbers of Subdivision Operations

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Outline

1 Motivation

2 The simplex case

- Barycentric subdivisions
- Edgewise subdivisions

3 Asymptotic behavior

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3 Asymptotic behavior

Notation

- **\mathbb{K} field**

- $A = \bigoplus_{i \geq 0} A_i$ standard graded \mathbb{K} -algebra

- **graded Betti numbers** of A :

$$\beta_{i,i+j}(A) := \dim_{\mathbb{K}} \text{Tor}_i^S(A, \mathbb{K})_{i+j}$$

- **Castelnuovo-Mumford regularity** of A :

$$\text{reg}(A) := \max\{j : \beta_{i,i+j}(A) \neq 0 \text{ for some } i\}$$

- **projective dimension** of A :

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Starting point

Study of **syzygies** of **Veronese** embeddings of algebraic varieties.

Recall: The r^{th} Veronese algebra of $A = \bigoplus_{i \geq 0} A_i$ is $A^{(r)} = \bigoplus_{i \geq 0} A_{ir}$.

Ein/Lazarsfeld:

- Considered the case $A = \mathbb{K}[x_1, \dots, x_n]$. Showed:

- For r sufficiently large

$$\beta_{i,i+j}(A^{(r)}) \neq 0 \text{ for all } 1 \leq j \leq n-1 \text{ and } i \in [a_j, b_j].$$

with endpoints a_j, b_j depending on j .

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$$\lim_{r \rightarrow \infty} \frac{\#\{i : \beta_{i,i+j}(A^{(r)}) \neq 0\}}{\text{pdim } A^{(r)}} = 1$$

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- Considered **Cohen-Macaulay** algebras A :

- ▶ Less explicit bounds for **non-vanishing** of $\beta_{i,i+j}(A^{(r)})$ (for $1 \leq j \leq \dim A - 1$).
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Question:

What can be said about the (**asymptotic**) behavior of the syzygies for Stanley-Reisner rings of (iterated) **barycentric** or **edgewise subdivisions** of a simplicial complex?

Navigation icons

Questions

Given:

Δ $(d - 1)$ -dimensional simplicial complex, $1 \leq j \leq d$

$\text{sd}^r(\Delta)$ iterated barycentric subdivision of Δ

$\Delta^{\langle r \rangle}$ r^{th} edgewise subdivision of Δ

- 1.) Let $\Delta = \Delta_{d-1}$ be the $(d - 1)$ -simplex. Which Betti numbers $\beta_{i,i+j}(\mathbb{K}[\text{sd}(\Delta_{d-1})])$ and $\beta_{i,i+j}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}])$, respectively, are non-zero?
- 2.) What happens asymptotically? I.e., for $r \rightarrow \infty$ study

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The barycentric subdivision

Δ simplicial complex

The **barycentric subdivision** of Δ is the simplicial complex $\text{sd}(\Delta)$ on vertex set $\Delta \setminus \{\emptyset\}$, whose faces are chains

$$\emptyset \neq A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_r$$

with $A_i \in \Delta \setminus \{\emptyset\}$ for $0 \leq i \leq r$.

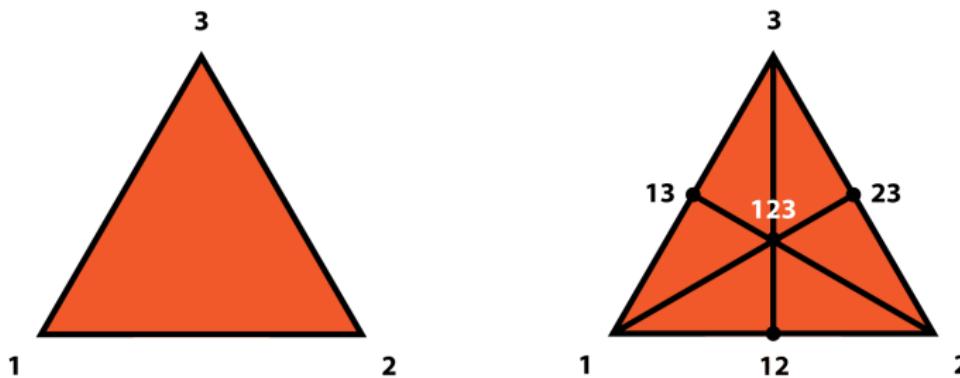
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Hochster's formula

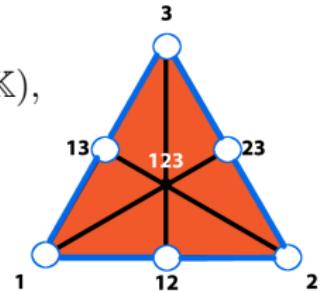
Δ simplicial complex on vertex set $[n] := \{1, 2, \dots, n\}$

$\mathbb{K}[\Delta]$ Stanley-Reisner ring

$$\beta_{i,i+j}(\mathbb{K}[\Delta]) = \sum_{\substack{W \subseteq [n] \\ \#W = i+j}} \dim_{\mathbb{K}} \tilde{H}_{j-1}(\Delta_W; \mathbb{K}),$$

where

$$\Delta_w = \{F \in \Delta : F \subseteq W\}.$$



In particular,

$$\beta_{i,i+j}(\mathbb{K}[\Delta]) \neq 0$$

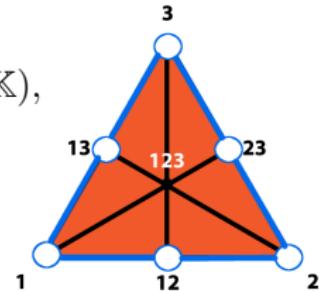
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The Castelnuovo-Mumford regularity of $\mathbb{K}[\text{sd}(\Delta)]$

$$\text{reg } \mathbb{K}[\text{sd}(\Delta)] = \begin{cases} \dim \Delta, & \text{if } \tilde{H}_{\dim \Delta}(\Delta; \mathbb{K}) = 0 \\ \dim \Delta + 1, & \text{if } \tilde{H}_{\dim \Delta}(\Delta; \mathbb{K}) \neq 0 \end{cases}$$

Proof:

First case: $\tilde{H}_{\dim \Delta}(\Delta; \mathbb{K}) = 0$

$$\Rightarrow \tilde{H}_{\dim \text{sd}(\Delta)}(\text{sd}(\Delta); \mathbb{K}) = 0$$

$\Rightarrow \tilde{H}_{\dim \text{sd}(\Delta)}(\text{sd}(\Delta)_W; \mathbb{K}) = 0$ for subsets W of the vertices of $\text{sd}(\Delta)$

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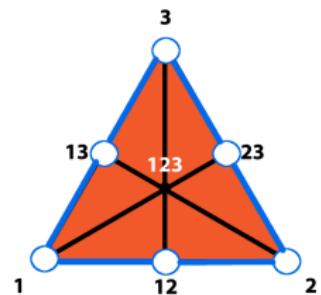
Let $d - 1 = \dim \Delta$.

Consider the boundary $\partial(F)$ of a $(d - 1)$ -face $F \in \Delta$.

$$\Rightarrow \text{sd}(\Delta)_{\{\emptyset \neq G \in \partial(F)\}} = \text{sd}(\partial(F)) \cong \partial(F) \cong \mathbb{S}^{d-2}$$

$$\Rightarrow \beta_{2^d - 2 - (d-1), 2^d - 2}(\mathbb{K}[\text{sd}(\Delta)]) \neq 0$$

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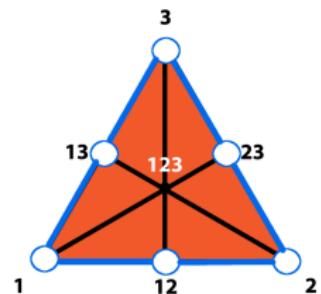
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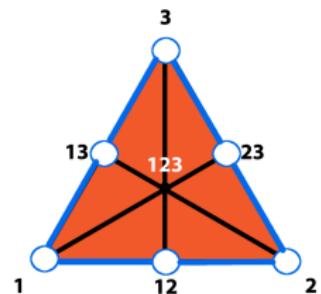
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Second case: $\tilde{H}_{\dim \Delta}(\Delta; \mathbb{K}) = 0$

- Note: $\text{reg } \mathbb{K}[\text{sd}(\Delta)] \leq \dim \text{sd}(\Delta) + 1 = \dim \Delta + 1$
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$$\tilde{H}_{\dim \text{sd}(\Delta)}(\text{sd}(\Delta); \mathbb{K}) \neq 0.$$

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The Castelnuovo-Mumford regularity of $\mathbb{K}[\text{sd}(\Delta)]$

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Proof:

Second case: $\tilde{H}_{\dim \Delta}(\Delta; \mathbb{K}) = 0$

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The barycentric subdivision of the simplex

Theorem (Conca, J., Welker)

Let $d \geq 1$ and Δ_{d-1} be the $(d-1)$ -simplex. Then:

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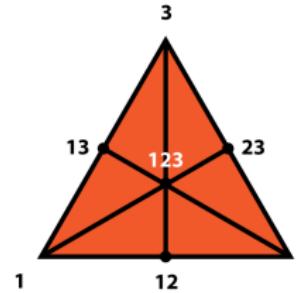
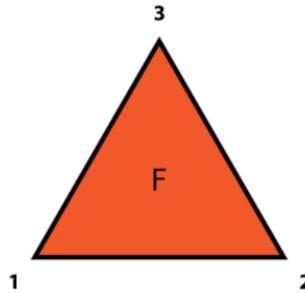
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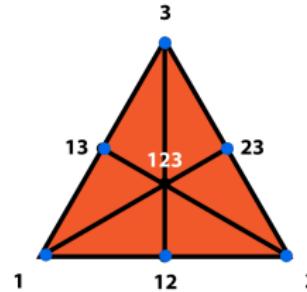
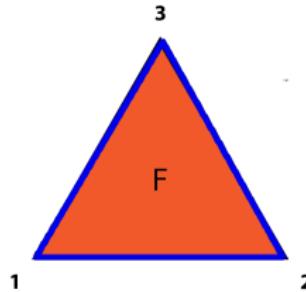
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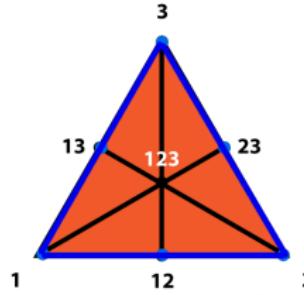
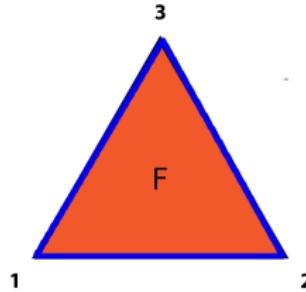
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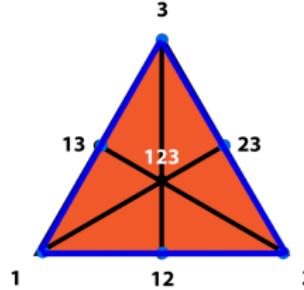
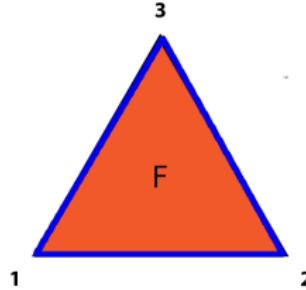
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Construct induced subcomplexes that are boundaries of cross polytopes.

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The $(j - 1)$ -dimensional cross polytope is the join of j 0-spheres.

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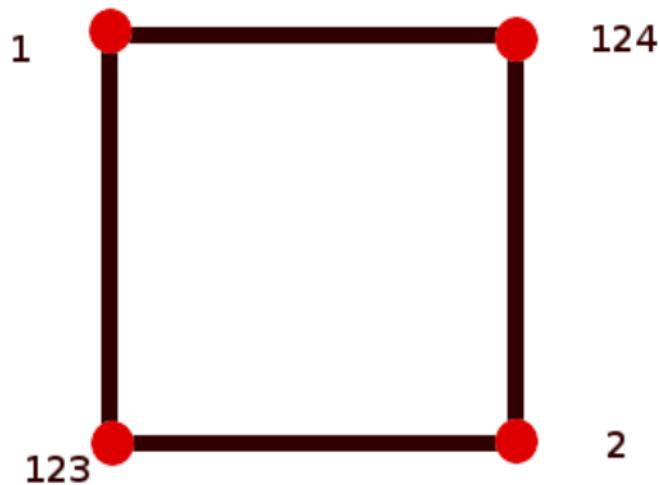
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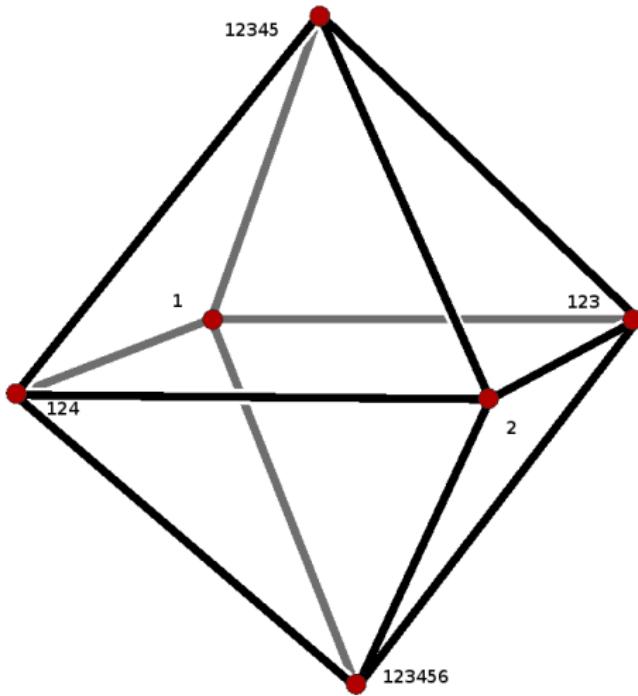
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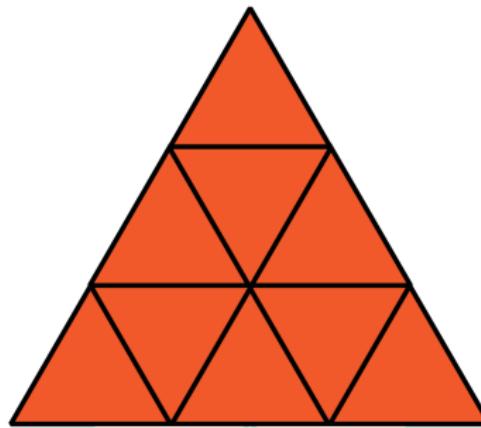
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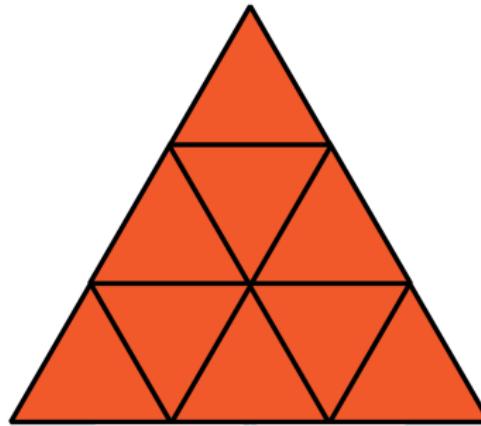
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- Basic idea: Edges are subdivided into r pieces.
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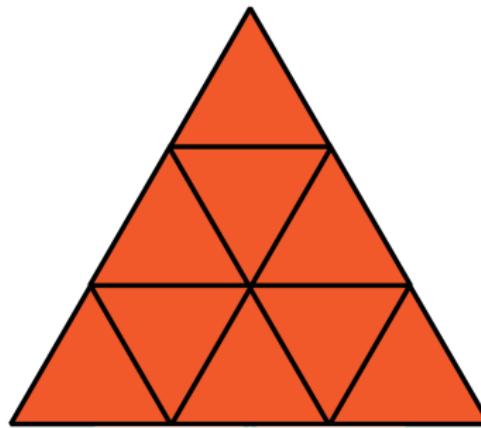
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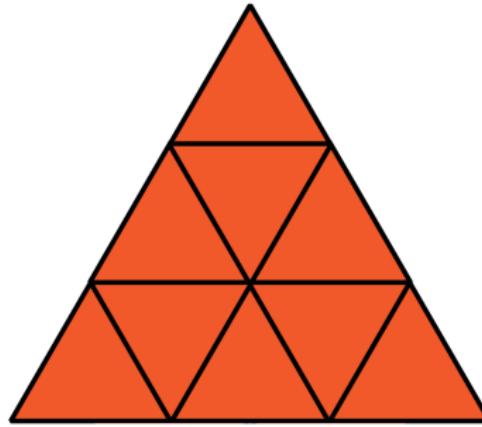
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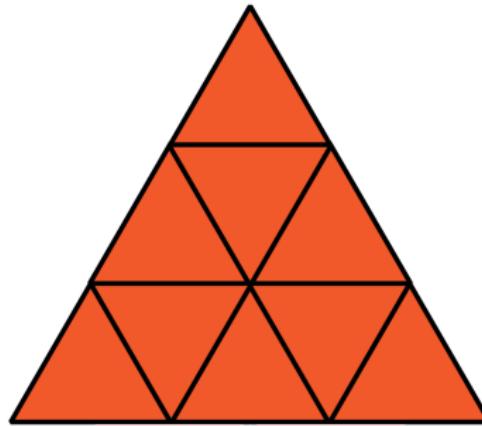
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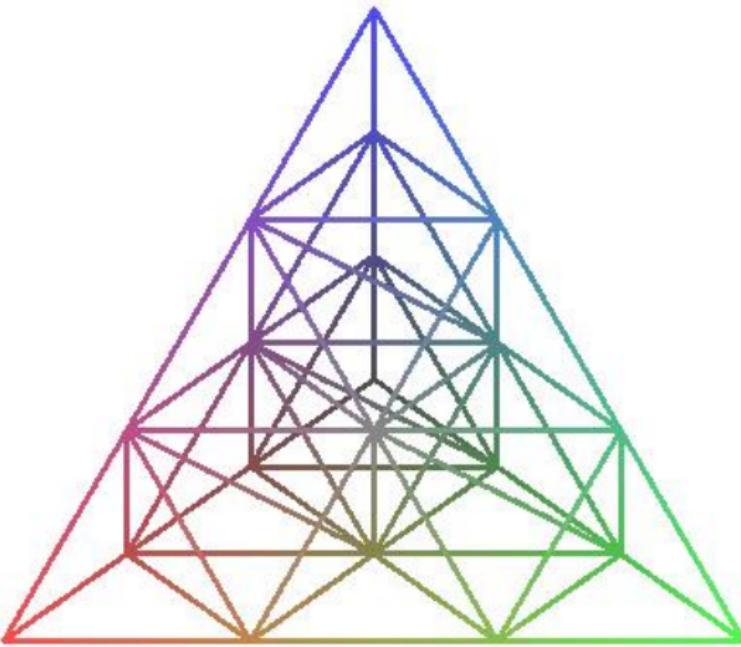


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The 3rd edgewise subdivision of the 3-simplex



From edgewise subdivisions to Veronese algebras

Proposition (Brun, Römer)

Δ simplicial complex on vertex set $[n]$, $r \geq 1$.

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Key observation:

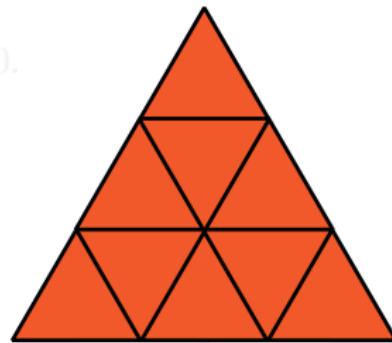
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Idea: Consider restrictions of this subcomplex.

- ⇒ If $\beta_{i,i+j}(\text{sd}(\Delta_{d-1})) \neq 0$, then $\beta_{i,i+j}(\Delta_{d-1}^{(r)}) \neq 0$.
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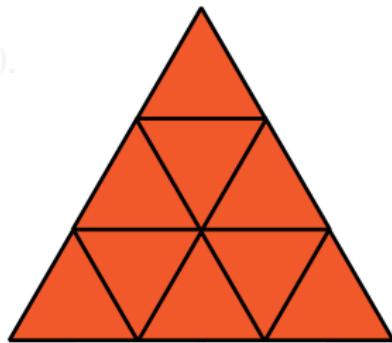
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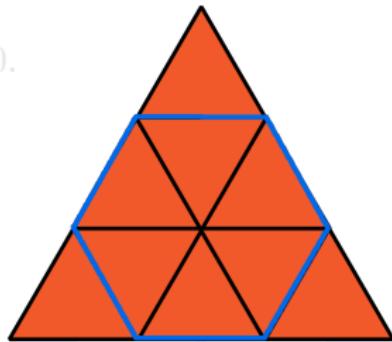
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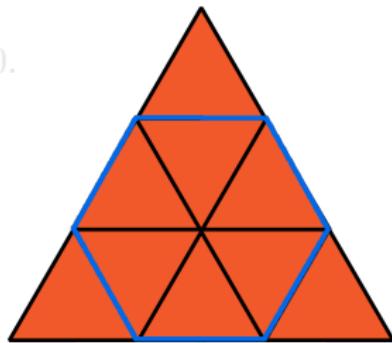
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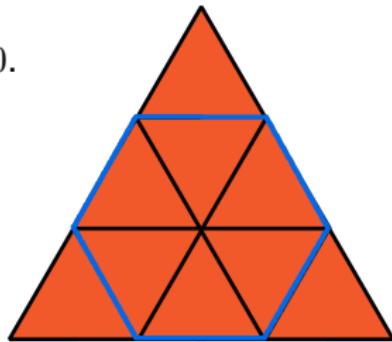
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Outline

1 Motivation

2 The simplex case

- Barycentric subdivisions
- Edgewise subdivisions

3 Asymptotic behavior

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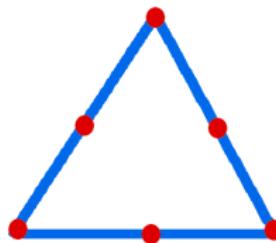
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Example:

- $\Delta = \partial(\Delta_d)$ boundary of the d -simplex
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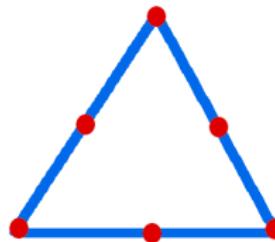
The special case $\tilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0$

Example:

- $\Delta = \partial(\Delta_d)$ boundary of the d -simplex
- $\text{sd}^r(\Delta)_W \cong \mathbb{S}^{d-1}$ if and only if W is the whole vertex set of $\text{sd}^r(\Delta)$.

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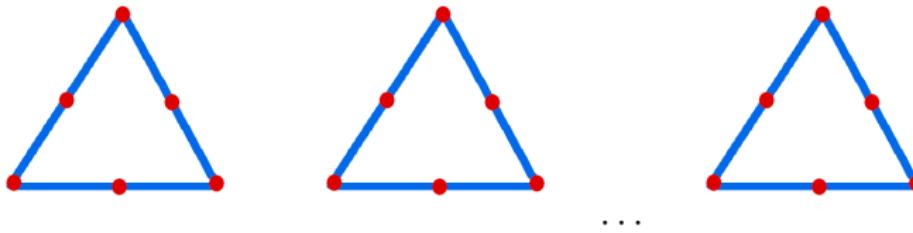
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Example:

- Δ disjoint union of m copies of $\partial(\Delta_d)$ with vertex sets V_1, \dots, V_m
- $\text{sd}^r(\Delta)_W \cong \mathbb{S}^{d-1}$ if and only if $2^{V_i} \setminus \{\emptyset\} \subseteq W$ for some i .

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$$\lim_{r \rightarrow \infty} \frac{\#\{i : \beta_{i,i+d}(\mathbb{K}[\text{sd}^r(\Delta)]) \neq 0\}}{\text{pdim } \mathbb{K}[\text{sd}^r(\Delta)]} = \frac{m-1}{m} = 1 - \frac{1}{m}$$



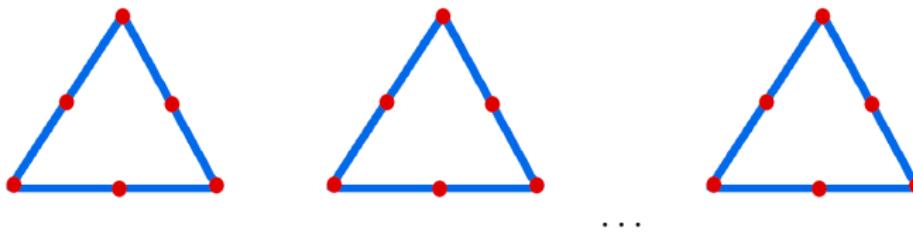
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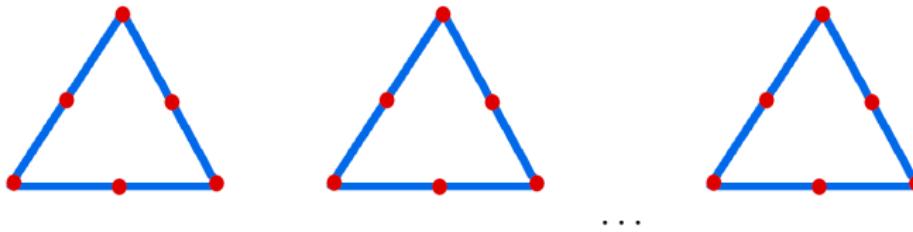
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For edgewise subdivisions we show that

$$\lim_{r \rightarrow \infty} \frac{\#\{i : \beta_{i,i+d}(\mathbb{K}[\Delta^{(r)}]) \neq 0\}}{\text{pdim } \mathbb{K}[\Delta^{(r)}]} = 1 - \frac{f_{d-1}^\sigma}{f_{d-1}^\Delta},$$

where

- σ is a **minimal $(d-1)$ -homology cycle**,
- f_{d-1}^Δ is the number of $(d-1)$ -faces of Δ ,
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The special case $\tilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0$

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Example:

- Let Δ_1 be a triangulation of \mathbb{S}^{d-1} with p facets.
- Let Δ_2 consist of q disjoint $(d-1)$ -simplices.
- $\Delta = \Delta_1 \cup \Delta_2$
- Then Δ_1 is a minimal homology $(d-1)$ -cycle and the limit is

$$1 - \frac{p}{p+q} = \frac{q}{p+q}.$$

In particular, any rational number $[0, 1]$ can occur as limit.

The special case $\tilde{H}_{d-1}(\Delta; \mathbb{K}) \neq 0$

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Thank you for your attention!