Test, multiplier and invariant ideals

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Measuring of Singularity

Multiplicity

Let f be a polynomial over a field $\mathbb{C}$, vanishing at $\mathbf{z} \in \mathbb{C}^n$.

We say f is *smooth* at $\mathbf{z}$ iff $\left.\dfrac{\partial f}{\partial x_i}\right|_{\mathbf{z}} \neq 0$ for some i.

For simplicity assume $\mathbf{z} = 0$.

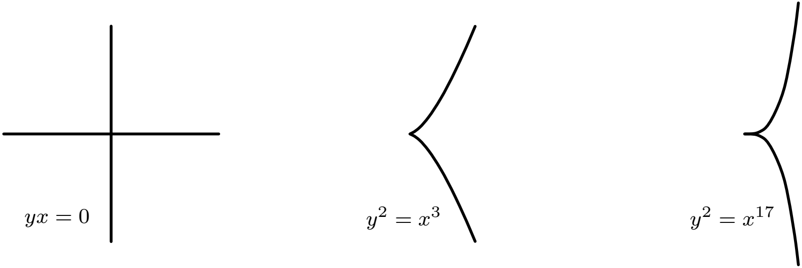
Define the *multiplicity*, or *order of singularity of f at $\mathbf{z}$* to be

$$\max\left\{d \,|\, \text{every differential } \partial \text{ of order } \leq d,\ \partial \cdot f|_{\mathbf{z}} = 0, \right\}$$

Measures of Singularity

Multiplicity

The multiplicity is an important first step in the study of singularities, but it is a coarse measure. The following curves of *multiplicity two* manifest different "levels" of singularity.



Test ideals in algebra, as well as *multiplier ideals* in birational geometry and analysis, spurred from an effort to better measure singularities.

Measuring Singularities

3 approaches - 2 characteristics - 2 invariants

These far more subtle measurements emerged from analytic, geometric and algebraic points of view.

- $\operatorname{char} K = 0$, the *log canonical threshold*, lct

  can be defined analytically (via integration), or geometrically (via *resolution of singularities*).

- $\operatorname{char} K = p > 0$, the F*-pure threshold*, fpt

  can be defined algebraically using the *Frobenius endomorphism*.

Remarkably, lct and fpt **define essentially the same invariant.**

"The smaller the values of these invariants, the worse the singularities."

Measuring Singularities

Analytic approach to the Log canonical threshold

$f \in \mathbb{C}[x_1, \ldots, x_n]$.

Analytically:

lct (f) = smallest real number $\lambda > 0$ s. t. $|f^{-2\lambda}|$ is not locally integrable

$$= \sup \left\{ \lambda \in \mathbb{R}_+ \;\middle|\; \int_{B_\varepsilon(0)} \frac{1}{|f|^{2\lambda}} < \infty, \text{ for some } \varepsilon > 0 \right\}.$$

Measuring Singularities

Analytic approach: $f = {z_1}^{a_1} \dots {z_n}^{a_n}$ **then** $\text{lct}(f) = \min_i \left\{ \frac{1}{a_i} \right\}$

Example

Let $f = {z_1}^{a_1} \dots {z_n}^{a_n}, \; a_i \in \mathbb{N}$.

Use polar coordinates to integrate $f \quad |z_i| = r_i, \; \det = r_1 \dots r_n$:

$$\int \frac{1}{|z_1|^{2a_1\lambda} \dots |z_n|^{2a_n\lambda}} \, d\mu = \int \frac{r_1 \dots r_n}{{r_1}^{2a_1\lambda} \dots {r_n}^{2a_n\lambda}} \, dr$$

Fubini's theorem:

$$\int_{B_\varepsilon(0)} \frac{1}{{r_1}^{2\lambda a_1 - 1} \dots {r_n}^{2\lambda a_n - 1}} \, dr < \infty \quad \text{for some } \varepsilon > 0$$

iff $\quad 2\lambda a_i - 1 < 1 \quad$ iff $\quad \lambda < \min_i \left\{ \frac{1}{a_i} \right\} = \text{lct}(f)$.

Measuring Singularities

Analytic approach: f is smooth at 0 then $\operatorname{lct}(f) = 1$.

Example

If f is smooth at 0 then f can be taken to be part of a system of local coordinates for $\mathbb{C}^n$ at the origin, thus

$$\int_{B_\varepsilon(0)} \frac{1}{|f|^{2\lambda}}\, d\mu < \infty \quad \text{iff} \quad \lambda < 1\,.$$

$$\operatorname{lct}(f) = 1$$

Measuring Singularities

Algebraic approach to the F-pure threshold

$f \in \mathbb{Z}_p[x_1, \ldots, x_n]$ where p is prime.

For $e \geq 0$, let

$$\nu_f(p^e) := \max\{r \geq 0 \mid f^r \notin (x_1{}^{p^e}, \ldots, x_n{}^{p^e})\}.$$

The F-pure threshold of f is

$$\text{fpt}(f) := \lim_{e \to \infty} \frac{\nu_f(p^e)}{p^e}.$$

the limit exists and is contained in $(0, 1] \cap \mathbb{Q}$.

$f = x^2 + y^3$

Example

Let $f \in \mathbb{Z}_2[x, y]$ with $p = 2$, and $f = x^2 + y^3$.

$$\nu_f(2) = \max\{r \geq 0 \,|\, (x^2 + y^3)^r \nsubseteq (x, y)^{[2]} = (x^2, y^2)\} = 0$$

$\dots$

$$\nu_f(2^3) = \max\{r \geq 0 \,|\, (x^2 + y^3)^r \nsubseteq (x, y)^{[8]} = (x^8, y^8)\} = 3 = 2^2 - 1$$

$$\nu_f(2^e) = \max\{r \geq 0 \,|\, (x^2 + y^3)^r \nsubseteq (x^{2^e}, y^{2^e})\} = 2^{e-1} - 1$$

$$\mathrm{fpt}\,(f_p) = \lim_{e \to \infty} \frac{2^{e-1} - 1}{2^e} = 1/2$$

Asymptotically F-threshold = log canonical threshold

Fix $f \in \mathbb{Z}[x_1, \ldots, x_n]$. $\rightsquigarrow$ compute $\operatorname{lct}(f)$

Or reduce modulo $p \rightsquigarrow f_p \in \mathbb{Z}_p[x_1, \ldots, x_n] \rightsquigarrow \operatorname{fpt}(f_p)$

Theorem (Hara-Yoshida 2003)

(1) For all primes p, $\operatorname{fpt}(f_p) \leq \operatorname{lct}(f)$;
(2) $\lim_{p \longrightarrow \infty} \operatorname{fpt}(f_p) = \operatorname{lct}(f)$.

Conjecture (Mustata-Takagi-Watanabe 2005)
Does equality holds for infinitely many primes?

$f = x^2 + y^3$

Example (Mustata-Takagi-Watanabe 2005)

Let $f = x^2 + y^3$.

$$f \in \mathbb{C}[x_1, \dots, x_n] \quad \rightsquigarrow \quad \mathrm{lct}\,(f) = \frac{5}{6}$$

$$f_p \in \mathbb{F}_p[x_1, \dots, x_n] \quad \rightsquigarrow \quad \mathrm{fpt}\,(f_p) = \begin{cases} 1/2 & \text{if } p = 2 \\ 2/3 & \text{if } p = 3 \\ 5/6 & \text{if } p \equiv 1 \mod 6 \\ 5/6 - \frac{1}{6p} & \text{if } p \equiv 5 \mod 6 \end{cases}$$

$$\lim_{p \longrightarrow \infty} \mathrm{fpt}\,(f) \;=\; 5/6 \;=\; \mathrm{lct}\,(f)\,.$$

There are infinitely many p for which $\mathrm{fpt}\,(f_p) \;=\; \mathrm{lct}\,(f)$. Work of Elkies shows that $\mathrm{fpt}\,(f_p) \;\neq\; \mathrm{lct}\,(f)$ for infinitely many primes.

f = elliptic curve

Example ([Bhatt, 2013])
Let f define an elliptic curve E in $\mathbb{P}^2$ over $\mathbb{Z}$.

$$f \in \mathbb{C}[x_1, \ldots, x_n] \quad \rightsquigarrow \quad \mathrm{lct}\,(f) = 1$$

$$f_p \in \mathbb{F}_p[x_1, \ldots, x_n] \quad \rightsquigarrow \quad E \in \mathbb{P}^2_{\mathbb{F}_p}$$

$$\rightsquigarrow \quad \mathrm{fpt}\,(f_p) = \begin{cases} 1 & \text{if } E \text{ is } \textit{ordinary}, \\ 1 - \frac{1}{p} & \text{if } E \text{ is } \textit{supersingular}. \end{cases}$$

There are infinitely many p for which $\mathrm{fpt}\,(f_p) = \mathrm{lct}\,(f)$. Work of Elkies shows that $\mathrm{fpt}\,(f_p) \neq \mathrm{lct}\,(f)$ for infinitely many primes.

The Frobenius endomorphism and $\mathrm{e^{th}}$-root

Let $S = K[x_1, \ldots, x_s]$ with $\mathrm{char}\, K = p \quad (K = \mathbb{Z}_p)$
and $\quad I = (f_1, \ldots, f_r)$ be an ideal. Fix $e \geq 0$, $q = p^e$.

Definition

(1) $\mathrm{F}^e \colon S \to S$, defined by $\mathrm{F}^e(g) = g^{p^e} = g^q$, for $g \in S$, denotes the e^{th} *Frobenius endomorphism* .

(2) The images of S and I are denoted
$S^q = \{g^q \mid g \in S\}$ and $I^{[q]} := \mathrm{F}^e(I) = (f_1^q, \ldots, f_r^q)$.

(3) One easily sees that

$$\mathcal{B}_e = \{ x_1^{u_1} \ldots x_s^{u_s} \mid \; 0 \leq u_1, \ldots, u_s \leq q-1\}$$

forms a basis for S over S^q

The e^{th}-root

Let $S = K[x_1, \ldots, x_s]$ with $\operatorname{char} K = p$, I an ideal, and fix $e \geq 0$ $q = p^e$.

Definition

The *e^{th}-root ideal of I, denoted $I^{[1/q]}$ to be the smallest ideal J such that $I \subseteq J^{[q]}$.*

Remark

If we express the generators of $I = (f_1, \ldots, f_s)$ in terms of $\mathcal{B}_e = \{\, x_1^{u_1} \ldots x_s^{u_s} \mid \ 0 \leq u_1, \ldots, u_s \leq q-1\}$ by

$$f_i = \sum_{\mu \in \mathcal{B}_e} g_{i\mu}^q \, \mu, \quad \text{for some polynomial } g_{i\mu}, \text{ for } i = 1, \ldots, s\,,$$

then," the q^{th}-root" of the coefficients generate $I^{[1/q]}$:

$$I^{[1/q]} = (\, g_{i\mu} \mid \mu \in \mathcal{B}\,,\ i = 1, \ldots, s\,)\,.$$

Generalized test ideals

Let I be an ideal and $e \geq 0, q$ as before.

Definition Given $\lambda \in \mathbb{R}_+$, the *generalized test ideal of I at λ* is

$$\tau(\lambda \bullet I) = \bigcup_{e \geq 0} \left(I^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} = \left(I^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \text{ for } e \gg 0\,.$$

Properties

It defines a non-increasing, right continuous family of ideals associated to I, in the sense that:

$$\begin{array}{lll} \forall\ \lambda \geq \lambda', & & \tau(\lambda \bullet I) \subseteq \tau(\lambda' \bullet I)\ ; \\ \forall\ \lambda & \exists\, \varepsilon > 0 : & \tau(\lambda \bullet I) = \tau(\lambda' \bullet I)\,,\ \ \forall\ \lambda' \in [\lambda, \lambda + \varepsilon)\,. \end{array}$$

Jumping numbers and F-threshold

Definition

The points of discontinuity, or "jumps", $\lambda \in \mathbb{R}_+$ for which
$\tau(\lambda \bullet I) \neq \tau(\lambda - \delta \bullet I)$ for all $\delta > 0$,
are called F-*jumping numbers of the ideal* I .
fpt $(I) := \min\{\lambda \in \mathbb{R}_+ \mid \tau(\lambda \bullet I) \neq S\} \equiv$ "first jump" .

Theorem (Blickle-Mustata-Smith 2008)

The set of jumping numbers is a discrete subset of $\mathbb{Q}$.

Multiplier ideal: Analytically

Definition Given $f \in \mathbb{C}[x_1, \ldots, x_N]$ vanishing at $\mathbf{z} \in \mathbb{C}^N$, the log-canonical threshold of f at $\mathbf{z}$ is defined as:

$$\operatorname{lct}_{\mathbf{z}}(f) = \sup\{\lambda \in \mathbb{R}_+ : \int_B \frac{1}{|f|^{2\lambda}} < \infty, \ \exists \text{ a ball } \mathbf{z} \in B\}$$

Example: $\operatorname{lct}_0(x_1^{a_1} \cdots x_N^{a_N}) = \min_i \{1/a_i\}.$

Definition Given an ideal $I = (f_1, \ldots, f_r) \subseteq S$ and $\lambda \in \mathbb{R}_+$, the *multiplier ideal with coefficient λ of I* is defined as

$$\mathcal{J}(\lambda \bullet I) := \left\{ g \in S : \frac{|g|}{(\sum_{i=1}^r |f_i|^2)^\lambda} \in L^1_{\mathrm{loc}} \right\},$$

where L^1_{loc} denotes the space of locally integrable functions.

Multiplier ideal: Geometrically

Definition Alternatively, given an ideal $I \subseteq S = K[x_1, \ldots, x_N]$, $\mathrm{char}(K) = 0$ and $\lambda \in \mathbb{R}_+$, the *multiplier ideal with coefficient λ of I* is

$$\mathcal{J}(\lambda \bullet I) = \pi_* \mathcal{O}_X(K_{X/\operatorname{Spec}(S)} - \lfloor \lambda \cdot F \rfloor)$$

where:

(i) $\pi : X \longrightarrow \operatorname{Spec}(S)$ is a log-resolution of the sheafication $\widetilde{I}$ of I;
(ii) $\pi^{-1}(\widetilde{I}) = \mathcal{O}_X(-F)$.
(iii) $K_{X/\operatorname{Spec}(S)}$ is the relative canonical divisor.

This simply means that X is non-singular, F is an effective divisor, the exceptional locus E of π is a divisor and $F + E$ has simple normal crossing support.

Log-resolutions like this, in characteristic 0, always exist by Hironaka's celebrated result.

The *log-canonical threshold* of an ideal $I \subseteq S$ is:

$$\mathrm{lct}\,(I) = \min\{\lambda \in \mathbb{R}_+ : \mathcal{J}(\lambda \bullet I) \neq S\}.$$

Multiplier ideals

Fix $I = (g_1, \ldots, g_n) \subseteq S = K[x_1, \ldots, x_N]$, $\mathrm{char}(K) = 0$.
If $g_i \in \mathbb{Z}$, $I \subseteq R$ is said to be reduced from characteristic zero to,

$$I_p = I \otimes_{\mathbb{Z}} \mathbb{Z}_p \subseteq \mathbb{Z}_p[x_1, \ldots, x_n]$$

in characteristic $p > 0$.

In characteristic zero, the *multiplier ideals of I* associate to I, a right continuous, non-decreasing family of ideals

$$\{\mathcal{J}(\lambda \bullet I)\}_{\lambda \geq 0},$$

where the first jumping number "defines" the *Log canonical threshold of the ideal I*, lct (I).

Asymptotically: Generalized test ideals = Multiplier ideals

Theorem (Hara-Yoshida 2003)

Assume that $I \subseteq S$ is an ideal reduced from characteristic zero to characteristic $p > 0$.

(1) For all λ and all primes p, $\tau(\lambda \bullet I) \subseteq \mathcal{J}(\lambda \bullet I)$;

(2) **Fixing** λ, one has $\tau(\lambda \bullet I) = \mathcal{J}(\lambda \bullet I)$ for $p \gg 0$.

Conjecture (Mustata-Takagi-Watanabe 2005)

Does equality hold (for all λ) for infinitely many primes?

Determinantal ideals

Given $1 \leq t \leq m \leq n$, consider the $m \times n$ matrix of indeterminates over a field K

$$\underline{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & \cdots & x_{mn} \end{pmatrix}.$$

Let $I = I_t(X_{m\times n}) \subseteq S = K[x_{11}, \ldots, x_{mn}]$, the ideal generated by t-minors of the $\underline{X}$.

Theorem (Johnson 2004, Miller-Singh-Varbaro 2013)

[Johnson] If $\operatorname{char} K = 0$,

$$\text{lct}\,(I) = \min\left\{ \frac{(n-k)(m-k)}{(t-k)} \;\middle|\; k = 0, \ldots, t-1 \right\}.$$

[MSV] If $\operatorname{char} K = p > 0$, $\text{fpt}\,(I) = \text{lct}\,(I)$ for all prime p.

Multiplier ideals of maximal determinantal ideals

Let $I_t = I_t(X_{m\times n}) \subseteq S = K[x_{11}, \ldots, x_{mn}]$ for all $1 \leq t \leq m$.

Theorem [Johnson, 2004]

Let $\mathrm{char}\, K = 0$. The multiplier ideal of I_m at λ, is

$$\mathcal{J}(\lambda \bullet I_m) = I_m^{\lfloor \lambda \rfloor - \mathrm{lct}\,(I) + 1}$$

for all $\lambda \geq \mathrm{lct}\,(I)$.

Theorem [Johnson, 2004]

Let $\mathrm{char}\, K = 0$ and $1 \leq t \leq m$. The multiplier ideal of I_t at λ satisfies

$$\mathcal{J}(\lambda \bullet I_t) = \bigcap_{i=1}^{m} I_i^{(\lfloor \lambda \cdot (t-i+1) \rfloor + 1 - (m-i+1)(n-i+1))}$$

for all $\lambda \geq \mathrm{lct}\,(I)$.

Test ideals of arbitrary determinantal ideals

Theorem[H.]
Let $R = k[x_1, \ldots, x_{mn}]$ $\operatorname{char} k = p > 0$ and $I = I_m(X_{m\times n})$.
For all $\lambda \geq \text{fpt}\,(I)$,

$$\tau\,(\lambda \bullet I\,) = I^{\lfloor\lambda\rfloor - \text{fpt}\,(I)+1} = \mathcal{J}(\lambda \bullet I) \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

(in all prime characteristics p) .
The set of F-jumping numbers of I is fpt $(I) + \mathbb{N}$.

Conjecture[H.-Varbaro]
For all $1 \leq t \leq m \leq n$ let $I = I_t(X_{m\times n})$,

$$\tau\,(\lambda \bullet I\,) = \mathcal{J}(\lambda \bullet I) \otimes_{\mathbb{Z}} \mathbb{Z}_p, \quad \text{for all } \lambda.$$

Notation

Given $1 \leq t \leq m \leq n$, consider the $m \times n$ matrix X of indeterminates over a field K. Let $K[X]$ be the polinomial ring in the entries of X and $I_t \subseteq K[X]$ be the prime ideal generated by the t-minors of X.

Fix two K-vector spaces, V and W, with $\dim V = m$ and $\dim W = n$,

and consider the group $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$.

The action of G on $K[X]$

Fixing basis for V and W, one can consider the action of G on $K[X]$ defined by

$$(A, B) \cdot X = AXB^{-1} \quad \forall\, (A, B) \in G = \mathrm{GL}(V) \times \mathrm{GL}(W).$$

With respect to this action, I_t is an invariant ideal.

In characteristic 0, De Concini, Eisenbud e Procesi [DEP] gave a description of the invariant ideals of $K[X]$.

Young diagrams

A *(Young) diagram* is a vector $\sigma = (\sigma_1, \ldots, \sigma_k)$ with positive integers as entries, such that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \geq 1$.

We write $\sigma = (r_1^{s_1}, r_2^{s_2}, \ldots)$ to denote the tuple with first s_1 entries equal to r_1, the following s_2 entries of σ are equal to r_2 and so on...

Given diagrams $\sigma = (\sigma_1, \ldots, \sigma_k)$ and $\tau = (\tau_1, \ldots, \tau_h)$, we write $\sigma \subseteq \tau$ if $k \leq h$ and $\sigma_i \leq \tau_i$ for all $i = 1, \ldots, k$.

Cauchy formula

Consider the natural diagonal action of G on $V \otimes W^*$, the following isomorphism is G-equivariant:

$$K[X] \cong \mathrm{Sym}(V \otimes W^*) = \bigoplus_{d \geq 0} \mathrm{Sym}^d(V \otimes W^*)$$

If $\mathrm{char}(K) = 0$, the *Cauchy formula* yields a decomposition of $\mathrm{Sym}(V \otimes W^*)$ into irreducible G-modules

$$\mathrm{Sym}(V \otimes W^*) \cong \bigoplus_{\sigma} S_\sigma V \otimes S_\sigma W^*,$$

where σ is a Young diagram with $\sigma_1 \leq m$ and S_σ denotes the Schur functor.

E.g., when $\sigma = (d)$, $S_{(d)} V = \bigwedge^d V$ and when $\sigma = (1^d) = (1, \dots, 1)$ $S_{(1^d)} V = \mathrm{Sym}^d V$.

The G-invariant ideals of $\mathrm{Sym}(V \otimes W^*)$.

Therefore, the G-invariant vector spaces of $\mathrm{Sym}(V \otimes W^*)$ correspond to such sets Σ of Young diagrams:

$$\Sigma \mapsto \bigoplus_{\sigma \in \Sigma} S_\sigma V \otimes S_\sigma W^*.$$

It is shown in [DEP] that such a vector space is an ideal iff:

$$\sigma \in \Sigma,\ \tau \supseteq \sigma \Rightarrow \tau \in \Sigma.$$

Further, the ideal I_σ generated by $S_\sigma V \otimes S_\sigma W^*$ admits the decomposition:

$$I_\sigma = \bigoplus_{\tau \supseteq \sigma} S_\tau V \otimes S_\tau W^*$$

Note: The determinantal ideal I_t corresponds to the ideal $I_{(t)}$.
Given a (finite) set Σ of Young diagrams, set:

$$I(\Sigma) = \sum_{\sigma \in \Sigma} I_\sigma.$$

γ-functions

Thus, in charateristic 0, the G-invariant ideals of $\mathrm{Sym}(V \otimes W^*)$ correspond to finite sets Σ of Young diagrams:

$$\Sigma \mapsto I(\Sigma).$$

For each $i \in \{1, \ldots, m\}$, define the γ_i function on Young diagrams by:

$$\gamma_i(\sigma) = \sum_{j=1}^{k} \max\{0, \sigma_j - i + 1\}, \quad \text{for} \quad \sigma = (\sigma_1, \ldots, \sigma_k).$$

Given a finite set of diagrams, Σ, we define the polytope $P_\Sigma \subseteq \mathbb{R}^m$ as the convex hull of $\{(\gamma_1(\sigma), \ldots, \gamma_m(\sigma)) : \sigma \in \Sigma\}$.

We give an explicit description of the multiplier ideals of $I(\Sigma)$ in terms of the polytope P_Σ.

The multiplier ideals of $I(\Sigma)$ when $\Sigma = \{\sigma\}$;

In this talk we focus on the description of the multiplier ideals of I_σ

In other words, we describe the multiplier ideals, and consequently the log canonical thresholds, of I_σ for each Young diagram σ.

In the case where $\sigma = (t)$, $I_\sigma = I_t$ is a determinantal ideal, we recover a result of [Johnson, 2003] using log-resolutions of determinantal varieties.
Recently, [DoCampo, 2012] recovered the formula for the log-canonical threshold of I_t from the study of jet schemes associated to determinantal varieties.

Given a product of minors $\Delta = \delta_1 \cdots \delta_k \in K[X]$, where δ_i is a α_i-minor of X, we define $\alpha = (\alpha_1, \ldots, \alpha_k)$ to be the shape of Δ.

We will see that the multiplier ideals of I_σ are generated by products of minors of prescribed shapes (described by γ-functions).

Theorem [H.-Varbaro]

Given a Young diagram $\sigma = (\sigma_1, \ldots, \sigma_k)$ with $\sigma_1 \leq m$, and $\lambda \in \mathbb{R}_+$, $\mathcal{J}(\lambda \bullet I_\sigma)$ is generated by products of minors whose shape α satisfies:

$$\gamma_i(\alpha) \geq \lfloor \lambda \gamma_i(\sigma) \rfloor + 1 - (m-i+1)(n-i+1) \ \ \forall\, i = 1, \ldots, m.$$

Equivalently,

$$\mathcal{J}(\lambda \bullet I_\sigma) = \bigcap_{i=1}^{m} I_i^{(\lfloor \lambda \gamma_i(\sigma) \rfloor + 1 - (m-i+1)(n-i+1))}$$

In particular, the log-canonical threshold of I_σ is given by

$$\operatorname{lct}(I_\sigma) = \min_i \left\{ \frac{(m-i+1)(n-i+1)}{\gamma_i(\sigma)} \right\}.$$

Proof

We work towards developing a theory that gives a description of the test ideals (hence the F-pure thresholds) of ideals with certain nice properties, in a polynomial over a field K of positive characteristic. We recover a

description of the multiplier ideals, from a result of [Hara-Yoshida, 2003]:

the test ideals at λ of the characteristic p reduction $\overset{=}{p \gg 0}$ characteristic p reduction of the multiplier ideals at λ

G-invariant ideals in positive characteristic

In positive characteristic, there isn't a characterization of the G-invariant ideals of $K[X]$.

A priori, there isn't an obvious way to define the ideals I_σ, in positive characteristic.

But we still know how to handle"enough" G-invariant ideals, even in positive characteristic, to do the job!

Given a Young diagram $\sigma = (\sigma_1, \ldots, \sigma_k)$ with $\sigma_1 \leq m$, we set

$$D_\sigma = I_{\sigma_1} \cdots I_{\sigma_k} \subseteq K[X].$$

Theorem[DEP] If $\mathrm{char}(K) = 0$, then $\overline{I_\sigma} = D_\sigma$.

Corollary: For each $\lambda \in \mathbb{R}_+$, $\mathcal{J}(\lambda \bullet I_\sigma) = \mathcal{J}(\lambda \bullet D_\sigma)$.

Test ideals

Recall/Definitions:

Let $S = K[x_1, \dots, x_N]$ be a polynomial ring over a field K, $\mathrm{char}(K) = p > 0$. Given an ideal $I = (f_1, \dots, f_r)$ and $q = p^e$, recall that:

(1)
$$I^{[q]} = (f_1^q, \dots, f_r^q) \subseteq S;$$

(2) $I^{[1/q]}$ denotes the smallest ($\exists!$) ideal $J \subseteq S$ for which $I \subseteq J^{[q]}$;

(3) the *test ideal of I at $\lambda (\in \mathbb{R}_+)$* is

$$\tau(\lambda \bullet I) = \bigcup_{e>0} \left(I^{\lceil \lambda q \rceil} \right)^{[1/q]}$$

Test ideals

Theorem[Hara-Yoshida]
Given $I \subseteq P = \mathbb{Z}[x_1, \dots, x_N]$ and $\lambda \in \mathbb{R}_+$, there exists a prime $p \gg 0$ such that:

$$\mathcal{J}(\lambda \bullet I \cdot P \otimes_{\mathbb{Z}} \mathbb{C}) \cdot P \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = \tau(\lambda \bullet I \cdot P \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z})$$

We computed the test ideals (hence F-pure thresholds) of every ideal of the form D_σ (products of determinantal ideals, cf. [H.-Varbaro])

It is worth noting that these "are the same" in all characteristics.

Recall that the F-pure threshold of determinantal ideals ($\sigma = (t)$) was known, cf. [MSV].

Big test ideals

Definition

Given an ideal $I \subseteq S = K[x_1, \dots, x_N]$ and $\mathfrak{p} \in \operatorname{Spec}(S)$, define the function $f_{I;\mathfrak{p}} : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ by: $f_{I;\mathfrak{p}}(s) = \max\{\ell : I^s \subseteq \mathfrak{p}^{(\ell)}\}$.

$f_{I;\mathfrak{p}}$ is linear, $f_{I;\mathfrak{p}}(s) = s\, f_{I;\mathfrak{p}}(1)$, so we set:

$$e_{\mathfrak{p}}(I) = f_{I;\mathfrak{p}}(1) = \max\{\ell : I \subseteq \mathfrak{p}^{(\ell)}\}.$$

Proposition If K has positive characteristic, then:

$$\tau(\lambda \bullet I) \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Spec}(S)} \mathfrak{p}^{(\lfloor \lambda e_{\mathfrak{p}}(I) \rfloor + 1 - \operatorname{ht}(\mathfrak{p}))} \quad \forall\ \lambda \in \mathbb{R}_+. \qquad (\star)$$

Definition An ideal $I \subseteq S$ has big test ideals if equality holds in $(\star)$, for all $\lambda \in \mathbb{R}_+$.

Condition $(\diamond)$

Naturally, for each $s \in \mathbb{Z}_{>0}$, one has:

$$I^s \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Spec}(S)} \mathfrak{p}^{(e_{\mathfrak{p}}(I)s)}.$$

To make this inclusion optimal we introduce the following condition:

Definition: An ideal *$I \subseteq S$ satisfies condition $(\diamond)$* if for all $\mathfrak{p} \in \operatorname{Spec}(S)$ there exists a function $g_{I;\mathfrak{p}} : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ for which:

$$\overline{I^s} = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(S)} \mathfrak{p}^{(g_{I;\mathfrak{p}}(s))} \quad \forall\ s \gg 0. \tag{1}$$

Remark: One easily sees that, for some $c \in \mathbb{N}$, one has

$$e_{\mathfrak{p}}(I)s - c \leq g_{I;\mathfrak{p}}(s) \leq e_{\mathfrak{p}}(I)s.$$

Condition $(\diamond+)$

A result of [Bruns] shows that, for every Young diagram σ, the ideal D_σ satisfies condition $(\diamond)$. Indeed:

$$\overline{D_\sigma^s} = \bigcap_{i=1}^{m} I_i^{(\gamma_i(\sigma)s)}$$

Definition: An ideal $I \subseteq S$ *satisfies condition* $(\diamond+)$ if it satisfies condition $(\diamond)$, and there exists a term order $\prec$ on S and a polynomial $F \in S$ such that:

(i) $\mathrm{in}_\prec(F)$ is a square-free monomial;

(ii) $F \in \mathfrak{p}^{(\mathrm{ht}(\mathfrak{p}))}$ for all $\mathfrak{p} \in \bigcup_{s \in \mathbb{Z}_{>0}} \mathrm{Ass}\left(\overline{I^s}\right)$.

Proposition For all Young diagram σ, the ideal D_σ satisfies $(\diamond+)$.

Theorem[H.-Varbaro]
If $I \subseteq S$ satisfies condition $(\diamond+)$, then it has big test ideals,

$$\text{i.e.} \qquad \tau(\lambda \bullet I) = \bigcap_{\mathfrak{p}\in \operatorname{Spec}(S)} \mathfrak{p}^{(\lfloor \lambda e_{\mathfrak{p}}(I)\rfloor + 1 - \operatorname{ht}(\mathfrak{p}))} \quad \forall\ \lambda \in \mathbb{R}_+. \qquad (\star)$$

Corollary $\tau(\lambda \bullet D_\sigma)$ is generated by products of minors whose shape α satisfies:

$$\gamma_i(\alpha) \geq \lfloor \lambda \gamma_i(\sigma) \rfloor + 1 - (m-i+1)(n-i+1) \ \ \forall\ i = 1, \dots, m.$$

Equivalently,

$$\tau(\lambda \bullet D_\sigma) = \bigcap_{i=1}^{m} I_i^{(\lfloor \lambda \gamma_i(\sigma)\rfloor + 1 - (m-i+1)(n-i+1))}.$$

In particular, the F-pure threshold is given by:

$$\operatorname{fpt}(D_\sigma) = \min_i \left\{ \frac{(m-i+1)(n-i+1)}{\gamma_i(\sigma)} \right\}.$$

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