Boij-Söderberg theory: Cones of homological invariants

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Graded modules and Betti numbers

$$S = k[x_1, \dots, x_n].$$

Always finitely generated graded modules.

M a graded module $\rightsquigarrow$ graded Betti numbers $\beta_{ij}(M)$.

$$\beta = \{\beta_{ij}(M)\} \in \mathbb{Q}^{[0,n]\times\mathbb{Z}}$$

The β generate a *positive* cone $C^{betti}(\mathrm{mod}, n)$ in $\mathbb{Q}^{[0,n]\times\mathbb{Z}}$.

Subcategories

mod is the category of *f.g. graded* S-modules.

$\mathcal{M}$ is an additive subcategory of mod.

Get subcone $C^{betti}(\mathcal{M}, n)$ of $C^{betti}(\text{mod}, n)$.

Example subcategories

Example (Of subcategories $\mathcal{M}$)

- CM^c, Cohen-Macaulay (CM) modules of codimension c.
- modArt_0, artinian modules generated in degree 0.
- $\mathrm{mod}_{0,m}$, modules generated in degree 0, m-regular.
- Sq, squarefree modules. (Natural module category of $\mathbb{N}^n$-graded modules containing Stanley-Reisner ideals and rings.)

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If $\mathcal{A}$ is an additive category, denote by $K\mathcal{A}$ the category of complexes of objects in $\mathcal{A}$.

Extremal rays

The cones are described by their extremal rays.

Theorem (Boij-Söderberg, Eisenbud-Schreyer)

The extremal rays in $C^{betti}(mod, n)$ are given by Betti diagrams of pure resolutions of CM modules

$$S(-d_0)^{\beta_{0,d_0}} \leftarrow \cdots \leftarrow S(-d_p)^{\beta_{p,d_p}},$$

$p \in [0, n]$. *All such pure resolutions exist.*

Cones of Hilbert functions

Hilbert function $h_j(M) = \dim_k M_j \rightsquigarrow H = \{h_j\} \in \mathbb{Q}^{\mathbb{N}}$.

$C^{\text{hilb}}(\mathcal{M}, n)$ subcone of $\mathbb{Q}^{\mathbb{N}}$ generated by Hilbert fuctions of modules M in $\mathcal{M}$.

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Theorem (M.Boij-G.Smith)

The extremal rays in $C^{hilb}(modArt_0, n)$ are given by Hilbert functions of $S/\langle x_1, \cdots, x_n\rangle^i$, $i \geq 1$.
The extremal rays of $C^{hilb}(mod_{0,m}, n)$ are also described.

Cones of cohomology tables

$\mathcal{F}$ coherent sheaf of dimension $\leq d$ on a projective space $\mathbb{P}$
$\rightsquigarrow$ graded cohomology $\gamma_{ij}(\mathcal{F}) = \dim_k H^i(\mathbb{P}, \mathcal{F}(j))$
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Such γ generate a subcone $C'(\mathrm{coh}_{\mathbb{P}}, d)$ of $\mathbb{Q}^{[0,d]\times\mathbb{Z}}$.

Regularity

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Consider complexes of coherent sheaves $\mathcal{F}^\bullet$ such that:

- $\mathcal{F}^\bullet$ is 1-regular
- The derived dual $\mathbb{R}\mathrm{Hom}_{\mathcal{O}_\mathbb{P}}(\mathcal{F}^\bullet, \omega_\mathbb{P})$ is $n+1$-regular. (This implies $\dim \mathrm{Supp}\, H^i(\mathcal{F}^\bullet) \leq n+1$.)

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$\rightsquigarrow$ Cohomology $\gamma = \{\gamma_{ij}\} \in \mathbb{Q}^{[0,n]\times\mathbb{Z}}$ and subcones

$$C^{\mathrm{cohom}}(\mathrm{coh}_\mathbb{P}, n) \subseteq C^{\mathrm{cohom}}(\mathrm{Kcoh}_\mathbb{P}, n) \subseteq \mathbb{Q}^{[0,n]\times\mathbb{Z}}.$$

Homological data

$F_\bullet$ a complex of *free* S-modules. It has three sets of homological invariants:

Homology:	$h_{ij} = \dim_k H_i(F_\bullet)_j$	$H = \{h_{ij}\}$
Betti:	$F_i = \oplus S(-j)^{\beta_{ij}}$	$B = \{\beta_{ij}\}$
Cohomology:	$c_{ij} = \dim_k H_i(\mathrm{Hom}(F_\bullet, \omega_S))_j$	$C = \{c_{ij}\}$

Note. $\omega_S = S(-n)$.

Cones of homological data

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- If CM_c is the subcone of KFreemod consisting of free resolutions of CM modules of codimension c, the projection

$$C^{\text{trip}}(CM^c, n) \xrightarrow{\cong} C^{betti}(CM^c, n)$$

is an isomorphism.

Cones

Describe these cones. What are the extremal rays?

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Nature of problem I

Resolutions of an ideal I.

- If $I = I_X$, Betti numbers reflect geometric properties of variety X: Clifford index for curves, Greens conjecture.
- If $I = I_\Delta$, Stanley-Reisner ideal, Betti numbers reflect combinatorial/homological properties of Δ.
- Similar for modules M which are "close" to I and of S/I, modules of low rank or degree.

Nature of problem II

Cones reflect what happens in the "limit" for Betti numbers, Hilbert functions, cohomology tables, as the "size" (i.e. degree/rank) of the module goes to infinitiy.

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An analog of this is stable homotopy theory in algebraic topology, where one consideres spectra, "limits" of $S^p \wedge X$ as $p \to \infty$ and the "limit" stable homotopy groups.

Ulrich sheaves

It is conjectured that every variety X of dimension d in a projective space, has an Ulrich sheaf, i.e. a coherent sheaf with the same cohomology table as $\mathcal{O}_{\mathbb{P}^d}$, up to a scalar multiple.

If this holds, the cones $C^{\text{cohom}}(\text{coh}_X, n)$ are the same for *all* embedded varieties X in a projective spaces.

Main objectives

Characterize the following cones, i.e. their extremal rays:

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Extremal rays

For graded modules or squarefree modules, the extremal rays for the cone of *Betti* diagrams, are given by Betti diagrams of *pure resolutions*:

$$S(-d_0)^{\beta_{0,d_0}} \leftarrow S(-d_1)^{\beta_{1,d_1}} \leftarrow \cdots \leftarrow S(-d_p)^{\beta_{p,d_p}}$$

where $0 \leq p \leq n$.

Conjecture on extremal rays

Homological triplets

Conjecture (Totally pure complexes)

In $C^{trip}(KFreeSq, n)$ the extremal rays are given by triplets (H, B, C) of pure free squarefree complexes

$$F_\bullet : S(-d_0)^{\beta_{0,d_0}} \leftarrow S(-d_1)^{\beta_{1,d_1}} \leftarrow \cdots \leftarrow S(-d_p)^{\beta_{p,d_p}}$$

such that:

I. *For every $p < q$ with $H_p(F_\bullet)$ and $H_q(F_\bullet)$ nonzero and $H_i(F_\bullet) = 0$ when $p < i < q$, then:*

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$$\min\{d \mid H_{q,d} \neq 0\} - KrulldimH_p \geq q - p + 1.$$

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(**Note.** This is also meaningful for $C^{trip}(\mathrm{KFreemod}, n)$.)

Dualities

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Then

- I. $\Leftrightarrow \mathbb{A} \circ \mathbb{D}(F_\bullet)$ being a pure free complex.
- II: $\Leftrightarrow (\mathbb{A} \circ \mathbb{D})^2(F_\bullet)$ being a pure free complex.
- Note that $(\mathbb{A} \circ \mathbb{D})^3 \cong \mathrm{Id}[-n]$.

The basic problems

Problem 1. Show existence of totally pure complexes.
Problem 2. Show they are exactly the extremal rays.

Connection between cones

A surprising connection

Theorem

There is an injections of cones

$$C^{cohom}(Kcoh_{\mathbb{P}}, n) \overset{\iota}{\hookrightarrow} C^{trip}(KFreeSq, n).$$

Moreover this injection comes from an algebraic association

$$\mathcal{F}^{\bullet} \overset{\hat{\iota}}{\mapsto} \mathbb{W}(\mathcal{F}^{\bullet}), \text{ a free squarefree complex}$$

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Conjecture

The map ι is an isomorphism of cones.

Connection between cones II

Problem 1 on the existence of totally pure complexes can then be transferred to a problem on the existence of certain complexes of coherent sheaves.

Existence I

Theorem

For each numerical triple (H, B, C) where H_p is nonzero only for one p, and C_p is nonzero for only one p, this totally pure free squarefree complex exists.

It is in fact nothing but a pure free resolution of a CM squarefree module.

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In characteristic zero they are the image by $\hat{\iota}$ of natural $GL(W)$-equivariant vector bundles on projective space $\mathbb{P}(W)$.

Restriction to vector bundles

Theorem (Essentially Eisenbud-Schreyer)

The restriction

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is an isomorphism of cones.

Existence goal

Show that for all possible *numerical* triplets (H, B, C) there does exist a free squarefree complex with these homological invariants.

Show that such complexes are in the image of $\hat{\iota}$.

Existence II

Theorem (F.-S.Sam)

For all numerically possible (H, B, C) with one nonzero homology module H_p, and with two nonzero cohomology modules C_q, there does exist a totally pure free squarefree complex.

The coherent sheaves which by $\hat{\iota}$ map to these complexes arise as equivariant coherent sheaves for a maximal parabolic subgroup $P \subseteq GL(W)$.

Conclusion

Characterize the following cones, i.e. their extremal rays:

1. $C^{\text{cohom}}(\text{Kcoh}_{\mathbb{P}}, n)$
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- Even if all totally pure complexes exists it is still only conjectural that they are *all* the extremal rays.
- Cone 2. is still uncharted, but various projections of subcones have been understood.