Frobenius splitting of matrix Schubert ideals and Gaussian conditional independence models

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Outline

1. Algebraic statistics. The source of our problem in Gaussian conditional independence models.
2. A few nice Frobenius splittings from Schubert calculus.
   - Unrestricted matrices via type A.
   - Symmetric matrices via type C.
3. Back to the statistics. Parametrized models.

Conditional independence

Algebraic statistics studies models, varieties of probability distributions, and their ideals.

A conditional independence (CI) statement has the form

$$X_A \perp\!\!\!\perp X_B \mid X_C$$

"X_A is independent of X_B, given X_C"

where X_A, X_B, X_C refer to subsets of variables in a distribution X.

(Have we found all the ways X_A and X_B can interact?)

Problem

Describe the implications among collections of CI statements.

That is: when is $$\mathcal{I}_{\mathcal{C}_1} + \mathcal{I}_{\mathcal{C}_2} = \bigcap_i \mathcal{J}_i$$

for $\mathcal{I}_{\mathcal{C}_1}, \mathcal{I}_{\mathcal{C}_2}$ ideals of CI statements and $\mathcal{J}_i$ tractable models?

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The discrete setting

If X is a vector of n discrete random variables, then models have ideals in $\mathbb{C}[x_{11\ldots 1}, \ldots, x_{r_1 r_2 \ldots r_n}]$, where $x_{i_1\ldots i_n} = \mathrm{Prob}(X_i = i_1, \ldots, X_n = i_n)$.

Let $D = [n] \setminus (A \cup B \cup C)$. The ideal of $X_A \perp\!\!\!\perp X_B \mid X_C$ is

$$\left\langle \left(\sum_{\ell \in D} x_{ijk\ell}\right)\left(\sum_{\ell \in D} x_{i'j'k\ell}\right) - \left(\sum_{\ell \in D} x_{i'jk\ell}\right)\left(\sum_{\ell \in D} x_{ij'k\ell}\right) : \right.$$

$$\left. i, i' \in \prod_{a \in A}[r_a],\ j, j' \in \prod_{b \in B}[r_b],\ k \in \prod_{c \in C}[r_c] \right\rangle.$$

Primary decomposition, Gröbner bases known for cases including

- [Fink] $\{X_1 \perp\!\!\!\perp X_2 \mid X_3, X_1 \perp\!\!\!\perp X_3 \mid X_2\}$
- [HHHKR, Ay-Rauh] $\{X_1 \perp\!\!\!\perp X_S \mid X_{[n]\setminus(1\cup S)} :$ various $S\}$, including binomial edge ideals
- [Swanson-Taylor] $\{X_i \perp\!\!\!\perp X_j \mid X_{[n]\setminus i,j} : 1 \leq i < j \leq t\}$ (all assoc primes, but only the minimal components)

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The Gaussian setting

An n-dimensional Gaussian random variable X with mean zero has the form $X = AW$, for $A \in \mathrm{GL}_n(\mathbb{R})$ and $W_i \sim \mathcal{N}(0, w_i)$ independent scalar Gaussian variables.

These distributions are characterised by their covariance matrices

$$\Sigma = A^T \operatorname{diag}(w) A$$

recording the covariances $\sigma_{ij} = \mathbf{E}(X_i X_j)$.
Σ is symmetric and positive (semi)definite.

So we work with ideals in $\mathbb{C}[\Sigma] := \mathbb{C}[\sigma_{11}, \dots, \sigma_{nn}]$, where $\sigma_{ij} = \sigma_{ji}$.

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Gaussian CI ideals

For $A, B \subseteq [n]$, $\quad X_A \perp\!\!\!\perp X_B$ is true when $\Sigma_{AB} = 0$, so

$$\mathcal{I}_{X_A \perp\!\!\!\perp X_B} = \langle \sigma_{ab} : a \in A, b \in B \rangle.$$

Conditioning on a variable reduces Σ to its Schur complement, so

$\mathcal{I}_{X_A \perp\!\!\!\perp X_B | X_C}$ is the determinantal ideal $\mathcal{I}_{1+\#C}(\Sigma_{A\cup C, B\cup C})$.

Our problem becomes

Problem

Decompose sums of determinantal ideals from submatrices of a single symmetric matrix.

Frobenius splittings

Recall: let $\mathbb{K}$ be a perfect field of char $p > 0$, R a $\mathbb{K}$-algebra. Then $\phi : R \to R$ is a Frobenius near-splitting if

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "\cdot a^p"'] & R \arrow[d, "\cdot a"] \\ R \arrow[r, "\phi"'] & R \end{tikzcd}$$

commutes for $a \in \mathbb{K}$, and a Frobenius splitting if further $f(1) = 1$.

An ideal $\mathcal{I} \subseteq R$ is compatibly split by ϕ if $\phi(\mathcal{I}) \subseteq \mathcal{I}$, i.e. if f descends to a splitting on $R/\mathcal{I}$.

The Frobenius near-splittings of $R = \mathbb{K}[x_1, \ldots, x_d]$ form an R-module isomorphic to R. Its generator, Tr, acts on monomials by

$$\mathrm{Tr}(m) = \begin{cases} (x_1 \cdots x_d m)^{1/p} / x_1 \cdots x_d & \text{if defined} \\ 0 & \text{else.} \end{cases}$$ [Brion-Kumar]

A source of Frobenius splittings over $\mathbb{Z}$

Say that $f \in S = \mathbb{Z}[x_1, \ldots, x_d]$ determines a Frobenius splitting if, for almost all p, $\mathrm{Tr}(f^{p-1}\cdot)$ is a Frobenius splitting of $\mathbb{F}_p[x_1, \ldots, x_d]$.

$\mathcal{I} \subseteq S$ is compatibly split by f if it's compatibly split by $\mathrm{Tr}(f^{p-1}\cdot)$ for almost all p.

Theorem ([Knutson '09])

Let $f \in \mathbb{Z}[x_1, \ldots, x_d]$ have degree d and initial term $x_1 x_2 \cdots x_d$ in lex order. Then f determines a Frobenius splitting.

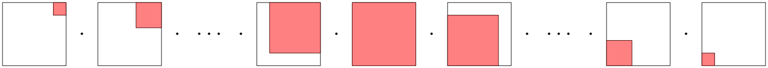
Let $\mathcal{J}$ be the smallest set of ideals containing $\langle f \rangle$ and such that

1. *If $I_1, I_2 \in \mathcal{J}$, then $I_1 + I_2, I_1 \cap I_2 \in \mathcal{J}$ and*
2. *If $I \in \mathcal{J}$ and J is a primary component of I then $J \in \mathcal{J}$.*

Then every ideal $J \in \mathcal{J}$ is compatibly split.
Over $\mathbb{Q}$, all such J are radical and have squarefree lex-initial ideal.

Example

In $\mathbb{Z}[x_{11}, \ldots, x_{nn}]$, let f_{full} be the product of the determinants of upper-right and lower-left-justified square submatrices of $[x_{ij}]$.



Theorem ([Knutson, Knutson-Lam-Speyer])

f_{full} *compatibly splits*

- *the matrix Schubert ideals;*
- *further ideals, to be described below.*

Nonstandard convention

Schubert cells are $\Omega_w = B^- \setminus B^- w B^-$, where B^- is lower-triangular matrices.
Upper-right rank conditions determine matrix Schubert ideals.

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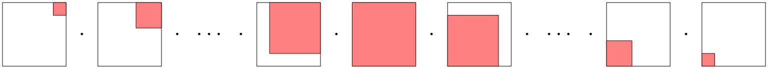

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Matrix Schubert ideals

Given a permutation matrix $w \in S_k$, let $R(w)$ be the $k \times k$ array

$$R_{ij} = \#\{(i',j') : i' \leq i, j' \geq j, w_{i',(n+1-j')} = 1\}.$$

Given a $k \times k$ array R of naturals, define $\mathcal{I}_R \subseteq \mathbb{C}[y_{11}, \ldots, y_{kk}]$ by

$$\mathcal{I}_R = \sum_{i,j} \mathcal{I}_{R_{ij}} \left(\begin{bmatrix} y_{1j} & & y_{1n} \\ & \iddots & \\ y_{ij} & & y_{in} \end{bmatrix} \right).$$

The matrix Schubert ideal $\mathcal{I}_w$ is $\mathcal{I}_{R(w)}$.

E.g. $w = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $R(w) = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 2 & 1 & 1 & \mathbf{1} \\ 3 & \mathbf{2} & 1 & 1 \\ 4 & 3 & \mathbf{2} & 1 \end{bmatrix}$, $\mathcal{I}_w \ni y_{12}, \begin{vmatrix} y_{23} & y_{24} \\ y_{33} & y_{34} \end{vmatrix}, \ldots$

Schubert varieties

Schubert varieties $\Omega_w = B^- \setminus B^- w B^-$ are, for the group SL_n, quotients of the $V(\mathcal{I}_w)$ by the lower triangular matrices B^- acting on the left.

As w varies, they form the closed cells of a stratification of $B^- \setminus \mathrm{SL}_n$. The Bruhat order is

$$v \leq w \Longleftrightarrow \Omega_v \supseteq \Omega_w.$$

The interiors Ω_w° of these cells are affine spaces.

Opposite Schubert varieties are $\Omega^w = B^- \setminus B^- w B^+$.

Rank conditions on enlarged matrices

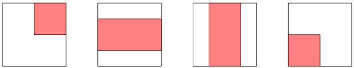
Let J be the $n \times n$ matrix with 1s on the antidiagonal and 0s off it. Define

$$Full := \left\{ \begin{bmatrix} J & x_{ij} \\ 0 & J \end{bmatrix} : x_{11}, \ldots, x_{nn} \in \mathbb{C} \right\},$$

and let $\mathcal{I}_{v,\mathrm{full}}$ be $full(\mathcal{I}_v)$, where
$full : \mathbb{C}[y_{11}, \ldots, y_{2n,2n}] \to \mathbb{C}[x_{11}, \ldots, x_{nn}]$ “evaluates $[y_{ij}]$ at *Full*”.

$\mathcal{I}_{R,\mathrm{full}}$ is generated by (top or left) and (bottom or right)-aligned

minors of $[x_{ij}]$.



Let $w_\square = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in S_{2n}$.

Full can be identified with $\Omega_\circ^{w_\square}$ so that intersecting with Ω_v gives

$$V(\mathcal{I}_{v,\mathrm{full}}) = Full \cap V(\mathcal{I}_v).$$

This is nonempty iff $v \leq w_\square$ in Bruhat order.

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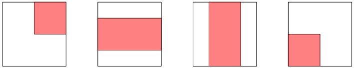

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This is nonempty iff $v \leq w_\square$ in Bruhat order.

All the compatibly split prime ideals

Proposition again (mostly [Knutson-Lam-Speyer])

The prime ideals compatibly split by f_{full} are exactly the $\mathcal{I}_{v,\text{full}}$ where $v \in [1, w_{\square}]$ in Bruhat order.

[Brion-Kumar] There are sheafy Frobenius splittings.
These are determined by certain divisors.

Compatibly split subvarieties of a variety with a splitting, and open subsets thereof, have induced splittings.

The full flag variety has a splitting f, compatibly splitting exactly the Richardson varieties $\Omega_v \cap \Omega^w$, with divisor $\sum$(codim 1 Richardsons).

$\Longrightarrow$ Check that $D(f_{\text{full}})$ agrees with the splitting induced from f.

The compatibly split ideals concretely

Proposition

The compatibly split ideals are exactly those of the form $\mathcal{I}_R$.

If $R_{i,j} \geq m := \min\{R_{i-1,j}+1, R_{i,j+1}+1, R_{i+1,j}, R_{i,j-1}\}$, then $R_{i,j}$ is redundant, and replacing it with m doesn't change $\mathcal{I}_R$.

Given two rank arrays R and R',

$$\mathcal{I}_R + \mathcal{I}_{R'} = \mathcal{I}_{\min\{R,R'\}}, \quad \mathcal{I}_R \cap \mathcal{I}_{R'} = \mathcal{I}_{\max\{R,R'\}}.$$

The primary decomposition of $\mathcal{I}_R$ is $\bigcap_i \mathcal{I}_{S_i}$ where the S_i are obtained from R by replacing subarrays

$$\begin{matrix} i+1 & i \\ i+1 & i+1 \end{matrix} \quad \text{with} \quad \begin{matrix} i & i \\ i+1 & i+1 \end{matrix} \quad \text{or} \quad \begin{matrix} i+1 & i \\ i+1 & i \end{matrix}$$

and decreasing redundant entries until neither is possible.

Variations

Parallel to f_{full}, define f_{up} and f_{sym} for upper-triangular and symmetric matrices of indeterminates. Set

$$Up := \left\{ \begin{bmatrix} J & x_{ij} \\ 0 & J \end{bmatrix} : [x_{ij}] \text{ is upper-triangular} \right\},$$

$$Sym := \left\{ \begin{bmatrix} J & x_{ij} \\ 0 & J \end{bmatrix} : [x_{ij}] \text{ is symmetric} \right\},$$

and $\mathcal{I}_{v,\mathrm{up}}$ and $\mathcal{I}_{v,\mathrm{sym}}$ be corresponding images of $\mathcal{I}_v$.

Proposition ([KLS], [FRS])

- *The compatibly split primes for f_{up} are the $\mathcal{I}_{v,\mathrm{up}}$ for $v \in [\begin{bmatrix} I & 0 \\ 0 & w_0 \end{bmatrix}, w_\square]$.*
- *The compatibly split primes for f_{sym} are the $\mathcal{I}_{v,\mathrm{sym}}$ for $v \in [1, w_\square]$ that commute with $w_0 := \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} \in S_{2n}$.*

Up and *Sym* as Schubert objects

Up itself equals $V(\mathcal{I}_{v,\mathrm{full}})$ for $v = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}$.

Sym is isomorphic to the type C opposite Schubert cell $\Omega_\circ^{w_\square,\mathrm{Sp}}$ in $\mathrm{Sp}_{2n}(\mathbb{C})/B^-$, where the symplectic form has matrix $E = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$.

Under the natural inclusion

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \mathrm{SL}_{2n}(\mathbb{C})^\sigma \stackrel{i}{\hookrightarrow} \mathrm{SL}_{2n}(\mathbb{C})$$

where $\sigma(A) = E(A^{-1})^\top E^{-1}$, Schubert varieties pull back to Schubert varieties, and *Full* to *Sym*. [Lakshmibai-Raghavan]

So $i^\#(\mathcal{I}_{v,\mathrm{full}}) = \mathcal{I}_{v,\mathrm{sym}}$ still defines the intersection of a Schubert variety with $\Omega_\circ^{w_\square,\mathrm{Sp}}$.

The symplectic Schubert varieties are indexed by $C_n \hookrightarrow S_{2n}$ whose image is elements commuting with w_0.

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The essential minors in $\mathcal{I}_{v,s}$ (for $s = \mathrm{full}$, $s = \mathrm{sym}$, $s = \mathrm{up}$) are those arising from non-redundant entries of $R(v)$.

Proposition ([Knutson], [Woo-Yong], [FRS])

In any term order selecting the diagonal terms from any minor, the essential minors in $\mathcal{I}_{v,\mathrm{full}}$ or $\mathcal{I}_{v,\mathrm{up}}$ form a Gröbner basis.

The same is true for $\mathcal{I}_{v,\mathrm{sym}}$ when it doesn't contain $\det(\Sigma)$.

The full and up cases are [WY] for one term order, involving pipe dreams.

Proposition

The number of compatibly split prime ideals for

- f_{full} *is*
$$\sum_{i=1}^{n+1} \left((i-1)!\,S(n+1,i)\right)^2$$
- f_{up} *is*
$$\sum_{i=1}^{n+1} (i-1)!\,S(n+1,i)$$

where $S(n,k)$ are Stirling numbers of the second kind, enumerating set partitions.

- f_{sym} *is the median Genocchi number H_{2n}:*

$i \setminus k$	0	1	2	3	4	5	6
1	**1**	1	**2**	2	**8**	8	**56**
2		1	1	3	6	14	48
3				3	3	17	34
4						17	17

(Each column is the partial sums of the previous, directions alternating.)

Algebraic-statistical translation

The CI statements $\mathcal{C}$ such that $\mathcal{I}_{\mathcal{C}}$ is compatibly split by f_{sym} are those that yield corner-aligned minors:

- $X_{[1,i]} \perp\!\!\!\perp X_{[j,n]}$, for $i < j$;
- $X_{[1,j-1]} \perp\!\!\!\perp X_{[i+1,n]} \mid X_{[j,i]}$, for $i \geq j$.

Theorem

Any sum of the above Gaussian CI ideals is an (explicit) intersection of prime sums of determinantal ideals.

All such sums have Gröbner bases composed of minors of Σ.

Gaussian graphical models

Let G be a directed acyclic graph on n vertices.

The *graphical model* of G is the semialgebraic set

$$\{((I-\Lambda)^{-1})^{\top}\operatorname{diag}(w)\,(I-\Lambda)^{-1} \ : \ \Lambda \in \mathbb{R}^{\text{adjacency(G)}}, w \in \mathbb{R}^m_{>0}\}$$

of positive definite matrices. Denote its ideal by $\mathcal{I}_G$.

$\mathcal{I}_G$ corresponds to a factorization

$$\mathrm{Prob}(X=x) = \prod_i \mathrm{Prob}(X_i = x_i \mid X_j = x_j : i \to j \in G).$$

$\mathcal{I}_G$ contains $\mathcal{I}_{x_A \perp\!\!\!\perp x_B | x_C}$ iff every undirected path from A to B either

- has a vertex $\ldots \leftarrow \bullet \rightarrow \ldots$ with no directed path to C; or
- meets C at another kind of vertex. [Koster]

$\mathcal{I}_G$ and $\sum \mathcal{I}_{x_A \perp\!\!\!\perp x_B | x_C}$ are equal up to saturation by $\langle \det(\Sigma) \rangle$.

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Comparing parametrizations

Suppose w fixes $n+1,\ldots,2n$, i.e. $\mathcal{I}_{w,\mathrm{full}}$ is a matrix Schubert ideal.

Note that $\mathcal{I}_{w,\mathrm{sym}} \not\ni \det(\Sigma)$.

Let $E_i(t) = I + te_{i,i+1}$ be Chevalley generators for SL_n.

Theorem

If $w|_{[n]} = s_{i_1}\cdots s_{i_k}$, then

$$\psi : (a_1,\ldots,a_n,t_1,\ldots,t_k) \mapsto \mathrm{diag}(a_1,\ldots,a_n)E_{i_1}(t_1)\cdots E_{i_k}(t_k)$$

parametrizes $V(\mathcal{I}_{w,\mathrm{up}})$.

Parametrizing symmetric matrix Schubert varieties

Proposition

Let $s : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ be the map $s(M) = M^\top M$. Then

$$V(\mathcal{I}_{w,\mathrm{sym}}) = \overline{s(V(\mathcal{I}_{w,\mathrm{up}}))}.$$

In particular, $V(\mathcal{I}_{w,\mathrm{sym}})$ is parametrized by $s \circ \phi$.

Key ingredient: $\mathcal{I}_{w,\mathrm{up}}$ is an initial ideal of $\mathcal{I}_{w,\mathrm{sym}}$.

Also:
s is generically 1–1 on $V(\mathcal{I}_{w,\mathrm{up}})$ by the Cholesky decomposition.
$\mathcal{I}_{w,\mathrm{sym}} \subseteq \mathcal{I}(s(V(\mathcal{I}_{w,\mathrm{up}})))$ is an easy direct check.

Generalised Markov chains

Which $\mathcal{I}_G$ have the form $\mathcal{I}_{w,\mathrm{sym}}$?

A directed acyclic graph G on the ordered set $[n]$ is a generalized Markov chain if for any edge $i \to j$ of G, there is also an edge $i' \to j'$ for all $i \leq i' < j' \leq j$.

G is a generalized Markov chain for some ordering $\Leftrightarrow$ the induced subgraph on the parents, resp. children, of any vertex is complete.

Theorem

$\mathcal{I}_G$ is compatibly split by f_{sym} $\Leftrightarrow$ G is a generalized Markov chain.

$\Rightarrow$ sums have good primary decompositions and Gröbner bases.

Thanks!

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