Koszul algebras and their syzygies
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## Free resolutions

$R$ standard graded $K$-algebra, i.e. $R=S / I$ and $S=K\left[x_{1}, \ldots, x_{n}\right]$
M f.g. graded $R$-module.
$\mathbb{F}_{M}$ minimal free resolution of $M$ as an $R$-module.
$\beta_{i, j}^{R}(M)=(i, j)$-th graded Betti number of $M$ as an $R$-mod
$i=$ homological degree, $j=$ internal degree
Is $\mathbb{F}_{M}$ finite?
Usually not.
Example of infinite resolution: $R=K[x] /\left(x^{n}\right)$ and $M=K$

$$
\ldots \longrightarrow R \xrightarrow{x^{n-1}} R \xrightarrow{x} R \xrightarrow{x^{n-1}} R \xrightarrow{x} R \longrightarrow 0
$$

## Auslander-Buchsbaum-Serre Theorem

Finite resolutions?
THM: The following conditions are equivalent
(1) Every $R$-mod has a finite free resolution.
(2) The residue field $K$ has a finite free resolution.
(3) $R$ is a polynomial ring ( $R$ is "regular").

The Koszul complex $K\left(m_{R}, R\right)$ of the max ideal $m_{R}$ of $R$ is the exterior algebra $\wedge \oplus_{i=1}^{n} R e_{i}$ with a DG $R$-algebra structure induced by differential

$$
\partial e_{i}=x_{i}
$$

$K\left(m_{S}, S\right)$ resolves $K$ over the polynomial ring $S$
Syzygies of modules over $S \leftrightarrow$ Koszul homology with coefficients in $M$

$$
\beta_{i j}^{S}(M)=\operatorname{dim}_{K} H_{i}\left(m_{S}, M\right)_{j}
$$

## Castelnuovo-Mumford regularity

How fast is the growth of the internal degrees of the syzygies as we increase their homological degree?

The (relative) Castelnuovo-Mumford regularity of $M$ is a measure of it

$$
\operatorname{reg}_{R}(M)=\sup \left\{j-i: \beta_{i j}^{R}(M) \neq 0\right\}
$$

Finite?
Not always: in the example $R=K[x] /\left(x^{n}\right)$ we have

$$
\operatorname{reg}_{R}(K)= \begin{cases}\infty & \text { if } n>2 \\ 0 & \text { if } n=2\end{cases}
$$

## Koszul algebras

(Priddy 1970) $R$ is Koszul if $\operatorname{reg}_{R} K=0$
i.e. the $i$-th syz module of $K$ as an $R$-mod are generated by elements of degree 1, i.e.

$$
\cdots \longrightarrow R(-3)^{*} \longrightarrow R(-2)^{*} \longrightarrow R(-1)^{*} \longrightarrow R \longrightarrow 0
$$

THM: (Avramov ${ }^{2}$-Eisenbud-Peeva) The following conditions are equivalent
(1) Every $R$ - $\bmod M$ has finite regularity, i.e. $\operatorname{reg}_{R} M \in \mathbb{Z}$.
(2) The residue field $K$ has finite regularity. i.e. $\operatorname{reg}_{R} K \in \mathbb{Z}$.
(3) $R$ is Koszul, i.e. $\operatorname{reg}_{R} K=0$.
$S=K\left[x_{1}, \ldots, x_{n}\right]$ Koszul
$K[x] /\left(x^{2}\right)$ is Koszul (and not regular)

## Quadrics and Gröbner bases of quadrics

$S=K\left[x_{1}, \ldots, x_{n}\right]$
$R=S / I$ is quadratic if $I$ is generated by quadrics
$R$ is G-quadratic if $I$ is generated by a Gröbner basis of quadrics. with respect to some system of coordinates and some term order $R$ is LG-quadratic: G-quadratic after possibly a lifting to higher dimension by regular sequence of linear forms. i.e. there exist a standard graded $K$-algebra $A$ and a regular sequence of linear forms $J=\left(L_{1}, \ldots, L_{u}\right) \subset A$ such that

$$
R \simeq A / J \text { and } A \text { is G-quadratic }
$$

## Facts

$\stackrel{\&}{\nLeftarrow} \stackrel{\notin}{\neq} \stackrel{\neq}{\neq}$ quadratic $\Rightarrow$ LG-quadratic $\Rightarrow$ Koszul $\Rightarrow$ quadratic

$$
\begin{gathered}
\Uparrow \\
\text { c.i. of quadrics }
\end{gathered}
$$

c.i. $=$ complete intersection

$$
R \text { Koszul } \Leftrightarrow \operatorname{Poinc}_{K}^{R}(z)=\frac{1}{\operatorname{Hilb}_{R}(-z)}
$$

$$
\operatorname{Poinc}_{K}^{R}(z)=\sum_{i \in \mathbb{N}} \beta_{i}^{R}(K) z^{i} \quad \operatorname{Hilb}_{R}(z)=\sum_{i \in \mathbb{N}}\left(\operatorname{dim} R_{i}\right) z^{i}
$$

## Example Quadratic and non-Koszul

$$
I=\left(a^{2}, b^{2}, c^{2}, d^{2}, a b+c d\right) \subset K[a, b, c, d]
$$

$$
\operatorname{Hilb}_{R}(z)=1+4 z+5 z^{2}
$$

$$
1 / \operatorname{Hilb}_{R}(-z)=1+4 z+\cdots+44 z^{5}-29 z^{6} \ldots
$$

$\operatorname{Poinc}_{R}(z)$ cannot be equal to $1 / \operatorname{Hilb}_{R}(-z)$
Numerical obstruction on the Hilbert series: no algebra with this HS can be Koszul

## Example LG-quadratic but not G-quadratic

A simple argument (of G.Caviglia) shows that any complete intersection of quadrics is LG-quadratic

Ex.: $\left(q_{1}, q_{2}, q_{3}\right)$ c.i. in variables x's.
$\left(q_{1}, q_{2}, q_{3}\right) \xrightarrow{\text { lift }}\left(y_{1}^{2}+q_{1}, y_{2}^{2}+q_{2}, y_{3}^{2}+q_{3}\right) \xrightarrow{\text { deform }}\left(y_{1}^{2}, y_{2}^{2}, y_{3}^{2}\right)$
$K[x] /\left(q_{1}, q_{2}, q_{3}\right)$ is LG-quadratic
(Eisenbud-Reeves-Totaro) $R$ is G-quadratic $\Rightarrow$ the ideal $/$ contains quadrics of "low" rank (Eisenbud-Reeves-Totaro)

It follows: 3 "general" quadrics in 3 variables define a
LG-quadratic algebra which is not G-quadratic

## Koszul and not LG-quadratic

$$
I=\left(a c, a^{2}-a d, b d, b c+a d, b^{2}\right)
$$

$R=S / I$ has HS

$$
\left(1+2 z-2 z^{2}-2 z^{3}+2 z^{4}\right) /(1-z)^{2}
$$

It is not LG-quadratic: no quadratic monomial ideal (in any number of var's) has that h-polynomial.

It is Koszul as can be shown using Koszul filtrations
This is the only example I know. An "HS unobstructed" example?
Candidate: $\left(a^{2}-b c, d^{2}, c d, b^{2}, a c, a b\right)$ is Koszul (by filtrations) and it is not HS-obstructed. That: there exist quadratic monomial ideals (in 5 or more var's) with that h-poly

## Classical algebras

Many classical algebras are G-quadratic in their "standard" coordinate system.

For instance coordinate rings of
Grassmannians, Schubert varieties, Flag varieties
Segre varieties, Veronese varieties, Hibi rings
Other examples of Koszul rings:
Coordinate rings of up to $2 n$ points in $\mathbb{P}^{n}$ in general position, of canonical curves (when quadratic)

Base rings of transversal polymatroids mentioned in M.Lason's talk in connection with White's conjecture

## Roos' family

Can we test Koszulness by computing the first $n$ or $2 n$ steps of the resolution of $K$ over $R$ ?

Roos: NO: there is a family of algebras $R_{u}$ in 6 variables depending on $u \in \mathbb{N}, u>2$, such that the first non-linear syzygy of $K$ over $R_{u}$ appears at step $u+1$.

No finite test for Koszulness

$$
R_{u}=\mathbb{Q}[a, b, c, d, e, f] / \begin{aligned}
& \left(a^{2}, a b, b^{2}, b c, c^{2}, c d, d^{2}, d e, e^{2}, e f, f^{2},\right. \\
& \\
& \\
& a c+u c f-d f, a d+c f+(u-2) d f)
\end{aligned}
$$

With $u=6$ the resolution of $\mathbb{Q}$ over $R_{u}$

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| total : | 1 | 6 | 28 | 120 | 496 | 2016 | 8128 | 32641 |
| $0:$ | 1 | 6 | 28 | 120 | 496 | 2016 | 8128 | 32640 |
| $1:$ | . | . | . | . | . | . | . | 1 |

## Important test example: Pinched Veronese

$$
P V(3,3,2)=K\left[x^{3}, y^{3}, z^{3}, x^{2} y, x^{2} z, x y^{2}, y^{2} z, x z^{2}, y z^{2}\right]
$$

$3=$ num.var., $3=\operatorname{deg}, 2=$ max cardinality of support.
Sturmfels and Peeva asked in 1996 whether $\operatorname{PV}(3,3,2)$ is Koszul (smooth toric variety defined by quadrics, no G-quad)
G.Caviglia shown in 2008 that $P V(3,3,2)$ is Koszul using a combination of various tools: deformations, filtrations, computer assisted computations

In 2011 with G.Caviglia we shown that $P V(3,3,2)$ is Koszul via an homological construction which allows us to classify the projections to $\mathbb{P}^{8}$ of the Veronese variety in $\mathbb{P}^{9}$ that are Koszul.

Thanh $V u$ (2013) generalizes this result: removing a single monomial from the Veronese algebra gives always a Koszul algebra apart from few well-known exceptions.

What about the others $P V(n, d, s)$ ?

## Syzygies of Koszul algebras

A Koszul algebra $R=S / I$ is quadratic.
It can be expressed by $\beta_{1 j}^{S}(R)=0$ for $j \neq 2$.
Other restrictions on $\beta_{i j}^{S}(R)$ imposed by the Koszul property?
This is a natural expectation because the resolution of $K$ over $R$ contains the Koszul complex $K\left(m_{R}, R\right)$ as a subcomplex (Tate construction and Gulliksen's Thm) and the homology of $K\left(m_{R}, R\right)$ gives the syzygies of $R$ over $S$.

Some potential properties are suggested by the monomial case via the Taylor resolution.

## Bounds of the $t_{i}$ 's

$R=S / I$. Set

$$
t_{i}^{S}(R)=\max \left\{j: \beta_{i j}^{S}(R) \neq 0\right\}=\max \left\{j: H_{i}\left(m_{R}, R\right)_{j} \neq 0\right\}
$$

In particular:
$t_{0}^{S}(R)=0$,
$t_{1}^{S}(R)=$ largest degree of a generator of $I$,
$t_{2}^{S}(R)=$ largest degree of a second $S$-syzygy of $R$ (first syzygy of l),

Taylor complex: If $I$ is generated by monomials of degree 2 (e.g.
$I=I_{\Delta}$ and $\Delta$ is flag complex) then $t_{i}^{S}(R) \leq 2 i$ for every $i$.
Corollary: If $R$ is LG-quadratic then $t_{i}^{S}(R) \leq 2 i$ for every $i$.

THM (Avramov-C-Iyengar, 2010) Assume $R$ is Koszul. Then
(1) $t_{i}(R) \leq 2 i$ for every $i$.
(2) if $t_{i}(R)<2 i$ for some $i$ then $t_{j}(R)<2 j$ for every $j \geq i$.
(3) $t_{i}(R)<2 i$ for $i>\operatorname{codim}(R)$.
(4) $\operatorname{reg}_{S}(R) \leq \operatorname{pd}(R)$ and $=$ holds iff $R$ is a c.i.

In 2013 we discover that Backelin proved (1) in 1988 in an unpublished manuscript and also that Kempf proved (1) in an article published in J.Alg. 1990 with title "Some wonderful rings in algebraic geometry".

Main point of our approach:

$$
\wedge^{i} H_{1}\left(m_{R}, R\right) \longrightarrow H_{i}\left(m_{R}, R\right)
$$

is surjective in (internal) degrees $\geq 2 i$

More generally one can ask if for a Koszul ring $R$ one has:

$$
t_{i+1}(R) \leq t_{i}(R)+2 \text { for every } i
$$

or even if

$$
t_{i+j}(R) \leq t_{i}(R)+t_{j}(R) \text { for every } i \text { and } j
$$

holds. Subaddittivity property for degree of syzygies.
THM (Avramov-C-lyengar, 2013) Assume $R$ is Koszul CM of characteristic 0 . Then
(1) $t_{i+j}(R) \leq \max \left\{t_{i}(R)+t_{j}(R), t_{i-1}(R)+t_{j-1}(R)+3\right\}$ all $i, j$.
(2) $t_{i+j}(R) \leq t_{i}(R)+t_{j}(R)+1$ all $i, j$.
(3) $t_{i+1}(R) \leq t_{i}(R)+2$ all $i$.
(4) If $R$ has the Green-Lazarsfeld $N_{p}$ property then

$$
\operatorname{reg}_{S}(R) \leq 2 \frac{\operatorname{pd}(R)}{p+1}+1
$$

$N_{p}=$ the syzygies on the quadrics are linear for $(p-1)$-steps.
Equivalently, $t_{i}(R)=i+1$ for $i=1, \ldots, p$
$N_{1}=$ quadratic, $N_{2}=$ quadratic and only linear syzygies on the quadrics
H.Dao in his talk presented a logarithmic bound for the regularity of a ring defined by monomials of degree 2 with the $N_{p}$ property $p>1$

Given $p>1$ there exist $a_{p}>1$ and $f_{p}(x) \in \mathbb{R}[x]$ such that

$$
\operatorname{reg}_{S}(R) \leq \log _{\mathrm{a}_{p}}\left(f_{p}(\operatorname{emb} \cdot \operatorname{dim}(R))\right.
$$

for every $R$ defined by monomials of degree 2 and with $N_{p}$ property
$R^{n}=$ the coordinate ring of the Grassmannian of $G(2, n)$. It is Koszul and has $\mathrm{N}_{2}$ in char 0 or large (Kurano) and

$$
\operatorname{reg}\left(R^{n}\right)=n-3 \quad \text { and } \quad \operatorname{emb} \cdot \operatorname{dim}\left(R^{n}\right)=\binom{n-2}{2}
$$

Hence: no logarithmic bound for the regularity of Koszul algebras with $N_{2}$ in terms of embdim
$R^{n, q}=q$-th Veronese in $n q$ variables, Koszul, $N_{q}$ (Green).... no logarithmic bound for the regularity of Koszul algebras with $N_{q}$ in terms of embdim.

Question: Does there exist a family $R^{n}$ with $n \in \mathbb{N}$ of Koszul algebras with $N_{2}$ such that reg and projdim (or embdim) are both linear in $n$ ?

Answer to Adam's question: For many families of classical ideals I (e.g. 2-minors of a $m \times n$ ), there is no initial ideal $J$ such that $\beta_{1}(I)=\beta_{1}(J)$, at least for high values of the parameters ( $m$ and $n$ in the example)

Main tool: splitting map of Koszul cycles (form Bruns-C-Römer)
A Koszul complex $K\left(y_{1}, \ldots, y_{n}, R\right)$ over a ring $R$. DG-algebra structure induces a multiplicative structure on the Koszul cycles $Z\left(y_{1}, \ldots, y_{n}, R\right)$. Set $Z_{a}=Z_{a}\left(y_{1}, \ldots, y_{n}, R\right)$

The multiplication map

$$
Z_{i} \otimes Z_{j} \longrightarrow Z_{i+j}
$$

factors as

$$
Z_{i} \otimes Z_{j} \xrightarrow{\alpha} Z_{i}\left(y_{1}, \ldots, y_{n}, Z_{j}\right) \xrightarrow{\beta} Z_{i+j}
$$

$\beta$ is surjective! (in char 0 or big)

Indeed

$$
Z_{i+j} \xrightarrow{\oplus} Z_{i}\left(y_{1}, \ldots, y_{n}, Z_{i}\right)
$$

(a direct summand) and $\beta$ is the projection back. This is the splitting map on Koszul cycles.
The Coker of $\alpha$ is

$$
T=\operatorname{Tor}_{1}^{R}\left(K_{i-1} / B_{i-1}, Z_{j}\right)
$$

Hence one obtains a surjective map:

$$
T \longrightarrow H_{i+j} / H_{i} * H_{j}
$$

It follows:

$$
t_{i+j}(R) \leq \max \left\{t_{i}(R)+t_{j}(R), \operatorname{reg}(T)\right\}
$$

The formulas in the THM are then obtained via bounds for the regularity $\operatorname{reg}(T)$.

Another application of the splitting map:
Thm (C-Murai) Let $I \subset S, \sqrt{I}=m_{S}$ and $Z_{i}=Z_{i}(I, S)$ (char 0 ). Then

$$
\operatorname{reg}_{S}\left(Z_{i+j}\right) \leq \operatorname{reg}_{S}\left(Z_{i}\right)+\operatorname{reg}_{S}\left(Z_{j}\right)
$$

Open questions: $R$ Koszul.

$$
\begin{align*}
& t_{i+j}(R) \leq t_{i}(R)+t_{j}(R) \text { every } i, j  \tag{??}\\
& \beta_{i}(R) \leq\binom{\beta_{1}(R)}{i} \text { every } i  \tag{??}\\
& \operatorname{pd}(R) \leq \beta_{1}(R) \tag{??}
\end{align*}
$$

Degree of the generators of Koszul homology?

That's all. Thanks for the attention!!! and MILLE GRAZIE to

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