

COMBINATORICS OF
SUBDIVISIONS
AND LOCAL h -VECTORS

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PLAN OF THE TALK

- A. Motivation
- B. h -vectors and local h -vectors
- C. Flag homology sphere and
 γ -vectors
- D. Examples

A. Motivation

A polynomial

$$f(x) = a_0 + a_1x + \dots + a_n x^n \in \mathbb{R}[x]$$

has palindromic (with center of symmetry $n/2$), nonnegative and unimodal coefficients if

- $a_i = a_{n-i}$
 - $a_i \geq 0$
 - $a_0 \leq a_1 \leq \dots \leq a_{\lfloor n/2 \rfloor}$.
- for $0 \leq i \leq n$

Examples (from combinatorics)

(a) Let

S_n = group of permutations
of $\{1, 2, \dots, n\}$

D_n = set of permutations $w \in S_n$
without fixed points

and for $w = (w_1, w_2, \dots, w_n) \in S_n$ let

$$\text{des}(w) = \# \{1 \leq i < n : w_i > w_{i+1}\}$$

$$\text{exc}(w) = \# \{1 \leq i \leq n : w_i > i\}$$

$$\text{inv}(w) = \# \{(i, j) : 1 \leq i < j \leq n, w_i > w_j\}.$$

$$(a1) \quad I_n(x) := \sum_{w \in S_n} x^{\text{inv}(w)}$$

$$= (1+x)(1+x+x^2) \cdots (1+x+\cdots+x^{n-1})$$

$$(a2) \quad A_n(x) := \sum_{w \in S_n} x^{\text{des}(w)} = \sum_{w \in S_n} x^{\text{exc}(w)}$$

$$= \begin{cases} 1, & n=1 \\ 1+x, & n=2 \\ 1+4x+x^2, & n=3 \\ 1+11x+11x^2+x^3, & n=4 \end{cases}$$

$$(a3) \quad d_n(x) := \sum_{w \in D_n} x^{\text{exc}(w)}$$

$$= \begin{cases} 0, & n=1 \\ x, & n=2 \\ x+x^2, & n=3 \\ x+\frac{7}{2}x^2+x^3, & n=4 \\ x+21x^2+21x^3+x^4, & n=5. \end{cases}$$

B. h -vectors and local h -vectors

Let

Δ = simplicial complex of dimension $n-1$

$f_i(\Delta)$ = # of i -dimensional faces.

Definition The h -polynomial of Δ is defined as

$$\begin{aligned} h(\Delta, x) &= \sum_{i=0}^n f_{i-1}(\Delta) x^i (1-x)^{n-i} \\ &= (1-x)^n f(\Delta, \frac{x}{1-x}) \\ &= \sum_{i=0}^n h_i(\Delta) x^i \end{aligned}$$

where

$$f(\Delta, x) = \sum_{i=0}^n f_{i-1}(\Delta) x^i.$$

(β) Let

$$\mathcal{B}_n = \{(w_1, w_2, \dots, w_n) : (|w_1|, |w_2|, \dots, |w_n|) \in \mathcal{G}_n\},$$

so that $\#\mathcal{B}_n = 2^n n!$.

(β₁)

$$\begin{aligned} (\beta_2) \quad \mathcal{B}_n(x) &:= \sum_{w \in \mathcal{B}_n} x^{\text{des}_{\mathcal{B}}(w)} \\ &= \begin{cases} 1+x, & n=1 \\ 1+6x+x^2, & n=2 \\ 1+23x+23x^2+x^3, & n=3 \end{cases} \end{aligned}$$

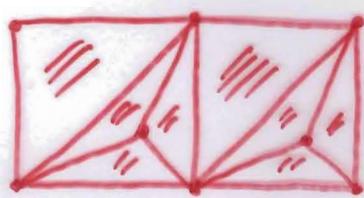
(β₃) ?

Question: can the unimodality of
these polynomials be interpreted
algebraic - geometrically ?

The h -vector of Δ is $h(\Delta) = (h_0(\Delta), \dots, h_n(\Delta))$.

Example

(a)

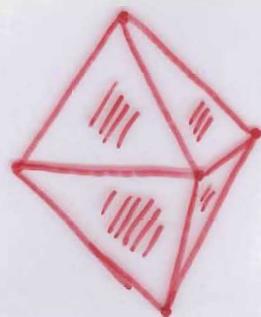


$$\|\Delta\| \approx B^2$$

$$f_0(\Delta) = 8, \quad f_1(\Delta) = 15,$$

$$f_2(\Delta) = 8$$

(b)



$$\|\Delta\| \approx S^2$$

$$(a) \quad f(\Delta, x) = 1 + 8x + 15x^2 + 8x^3$$

$$h(\Delta, x) = (1-x)^3 + 8x(1-x)^2 + 15x^2(1-x)$$

$$+ 8x^3$$

$$= 1 + 5x + 2x^2$$

$$(b) \quad f(\Delta, x) = 1 + 5x + 9x^2 + 6x^3$$

$$h(\Delta, x) = 1 + 2x + 2x^2 + x^3$$

$$= (1+x)(1+x+x^2).$$

Remark

$$\left\{ \begin{array}{l} h_0(\Delta) = 1 \\ h_1(\Delta) = f_0(\Delta) - n \\ h_n(\Delta) = (-1)^{n-1} \sum_{i \geq 0} (-1)^{n-i-1} f_{i-1}(\Delta) \\ \quad = (-1)^{n-1} \tilde{\propto} (\|\Delta\|) \end{array} \right.$$

and

$$h_0(\Delta) + h_1(\Delta) + \cdots + h_n(\Delta) = f_{n-1}(\Delta).$$

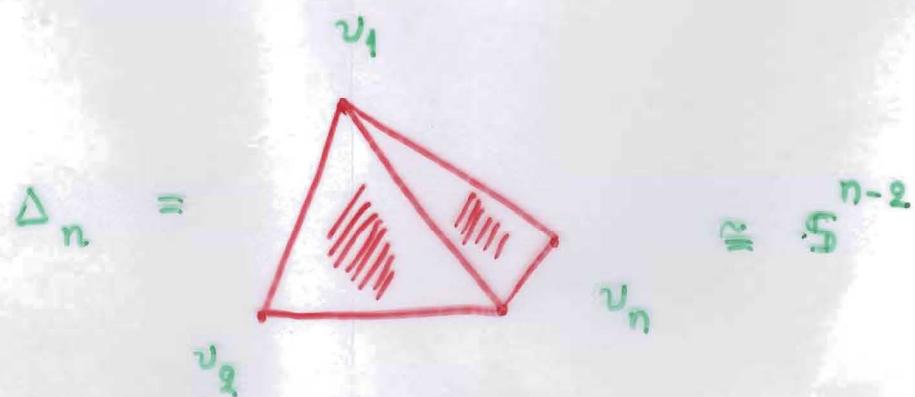
Theorem (Klee, Hochster, Reisner, Stanley)

The polynomial $h(\Delta, x)$ has

- palindromic coefficients if Δ is Eulerian (e.g. if $\|\Delta\| \leq S^{n-1}$).
- nonnegative coefficients if Δ is Cohen-Macaulay (e.g. if $\|\Delta\| \leq B^{n-1}$ or S^{n-1}).
- unimodal coefficients if Δ is the boundary complex of a simplicial n -dimensional polytope.

Examples

(a1) If

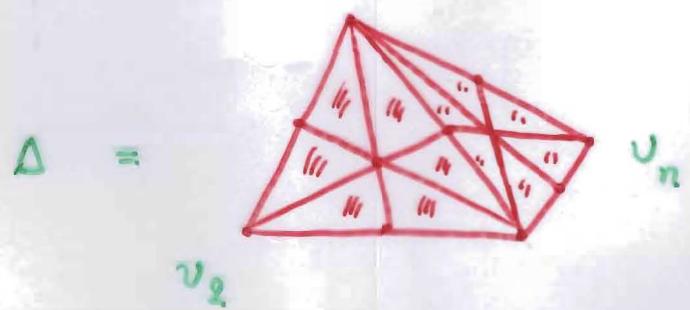


then $h(\Delta_n, x) = 1 + x + x^2 + \cdots + x^{n-1}$.

If $\Delta = \Delta_2 * \Delta_3 * \cdots * \Delta_n$, then

$$\begin{aligned} h(\Delta, x) &= (1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}) \\ &= I_n(x). \end{aligned}$$

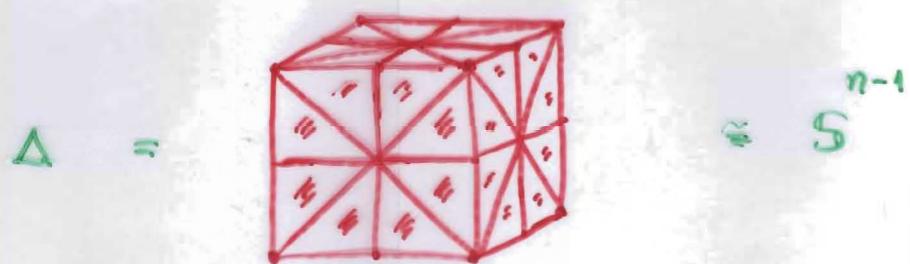
(αΩ) If



then

$$h(\Delta, x) = \sum_{w \in G_n} x^{\text{des}(w)} = A_n(x).$$

(βΩ) If



then

$$h(\Delta, x) = \sum_{w \in B_n} x^{\text{des}_B(w)} = B_n(x).$$

Let

- $V = n$ -element set
- $2^V =$ simplex on vertex set V
- $\Gamma =$ simplicial subdivision (triangulation) of 2^V .

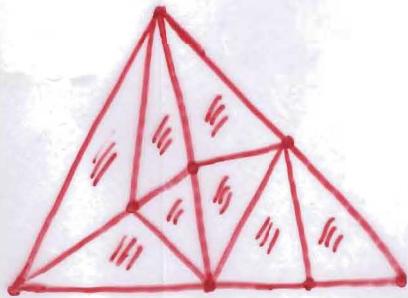
Definition (Stanley, 1992) The local h -polynomial of Γ (with respect to V) is defined as

$$\begin{aligned} e_V(\Gamma, x) &= \sum_{i=0}^n e_i(\Gamma) x^i \\ &= \sum_{F \subseteq V} (-1)^{|V|-|F|} h(\Gamma_F, x) \end{aligned}$$

where Γ_F is the restriction of Γ to the face F of the simplex 2^V . The sequence $e_V(\Gamma) = (e_0(\Gamma), e_1(\Gamma), \dots, e_n(\Gamma))$ is the local h -vector of Γ .

Example If

$$\Gamma =$$



then

$$\begin{aligned} e_V(\Gamma, x) &= (1+5x+2x^2) - (1+2x) \\ &\quad - (1+x) - 1 + 1 + 1 + 1 - 1 \\ &= 2x + 2x^2 \end{aligned}$$

Remark

$$e_0(\Gamma) = e_n(\Gamma) = 0$$

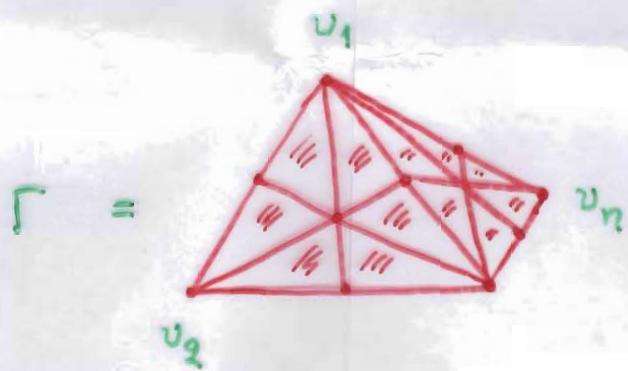
$e_1(\Gamma) = e_{n-1}(\Gamma) = \# \text{ of interior vertices}$
 $\text{of } \Gamma$

Theorem (Stanley, 1999)

The polynomial $\ell_V(\Gamma, x)$ has

- palindromic and nonnegative coefficients
- unimodal coefficients, if Γ is regular

Example If



then

$$\begin{aligned}\ell_V(\Gamma, x) &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} A_k(x) \\ &= \sum_{w \in D_n} x^{\text{exc}(w)} = d_n(x).\end{aligned}$$

Theorem (Stanley, 1992)

For every pure simplicial complex Δ and every triangulation Δ' of Δ

$$h(\Delta', x) = h(\Delta, x) + \sum_{F \in \Delta \setminus \{\emptyset\}} \epsilon_F'(\Delta'_F, x) h(\text{lk}_{\Delta}(F), x)$$

where $\text{lk}_{\Delta}(F)$ is the link of Δ at the face F .

Corollary $h(\Delta', x) \geq h(\Delta, x)$ coefficient-wise if Δ is Buchsbaum (e.g. if $\|\Delta\|$ is a manifold).

The theory generalizes to

- polyhedral (and more general)
subdivisions and toric h -vectors
(Stanley, 1992)
- cubical subdivisions and cubical
 h -vectors (A, 2012)

C. Flag spheres and γ -vectors

A polynomial $f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{R}[x]$ has palindromic coefficients ($a_i = a_{n-i}$ for $0 \leq i \leq n$) if and only if

$$f(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$$

for some (uniquely defined) $\gamma_0, \gamma_1, \dots, \gamma_{\lfloor n/2 \rfloor} \in \mathbb{R}$.

Definition The γ -polynomial associated to $f(x)$ is $\gamma(x) = \gamma_0 + \gamma_1x + \dots + \gamma_{\lfloor n/2 \rfloor} x^{\lfloor n/2 \rfloor}$. We say that $f(x)$ is γ -nonnegative if $\gamma_i \geq 0$ for every i .

Example If $f(x) = 1 + 6x + 13x^2 + 6x^3 + x^4$,

then

$$\begin{aligned}f(x) &= (1+x)^4 + 2x + 7x^2 + 2x^3 \\&= (1+x)^4 + 2x(1+x)^2 + 3x^2\end{aligned}$$

and hence

$$g(x) = 1 + 2x + 3x^2.$$

Remark $f(x)$ \geq -nonnegative \Rightarrow

$f(x)$ has (palindromic,) nonnegative
and unimodal coefficients.

Example The following polynomial are \$g\$-nonnegative:

- $A_n(x)$ (Foata-Schützenberger, 1970)
- $B_n(x)$ (Petersen, 2007)
- $d_n(x)$ (AS, GSW, SZ, LSW)

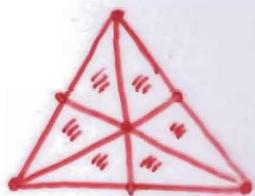
but not

$$\begin{aligned} I_3(x) &= (1+x)(1+x+x^2) \\ &= 1+2x+2x^2+x^3 \\ &= (1+x)^3 - x(1+x). \end{aligned}$$

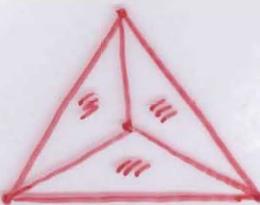
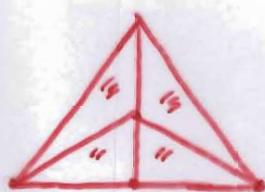
Remark $I_3(x) = h(\Delta, x)$ for

$$\Delta = \begin{array}{c} \text{Diagram of a cube with diagonal lines from vertices to opposite corners, representing the set } \Delta. \end{array} \cong S^2$$

Definition Δ is called flag if every minimal nonface of Δ has two elements.



flag



not flag

Conjecture 1: (Gal, 2005, Charney - Davis)

$h(\Delta, x)$ is γ -nonnegative for every flag triangulation Δ of the sphere.

Theorem (Davis - Okun, 2001, Gal) True

for $\dim(\Delta) \leq 4$.

Let

$$\gamma(\Delta, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i(\Delta) x^i$$

be the γ -polynomial associated to the h -polynomial $h(\Delta, x)$ of a triangulation Δ of S^{n-1} , so that

$$h(\Delta, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i(\Delta) x^i (1+x)^{n-2i}.$$

Conjecture 2 (Postnikov - Reiner - Williams, 2008) We have $\gamma(\Delta', x) \geq \gamma(\Delta, x)$ coefficientwise for all flag triangulations Δ' , Δ of S^{n-1} for which Δ' subdivides Δ .

Theorem (A, 2012) True for $\dim(\Delta) \leq 4$.

Let

$$\begin{cases} V = n\text{-element set} \\ \Gamma = \text{triangulation of } \Delta^V. \end{cases}$$

and

$$F_V(\Gamma, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} F_i(\Gamma) x^i$$

be the γ -polynomial associated to the local n -polynomial $\ell_V(\Gamma, x)$, so that

$$\ell_V(\Gamma, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} F_i(\Gamma) x^i (1+x)^{n-2i}.$$

Proposition For triangulations Δ', Δ' of Δ^{n-1} for which Δ' subdivides Δ

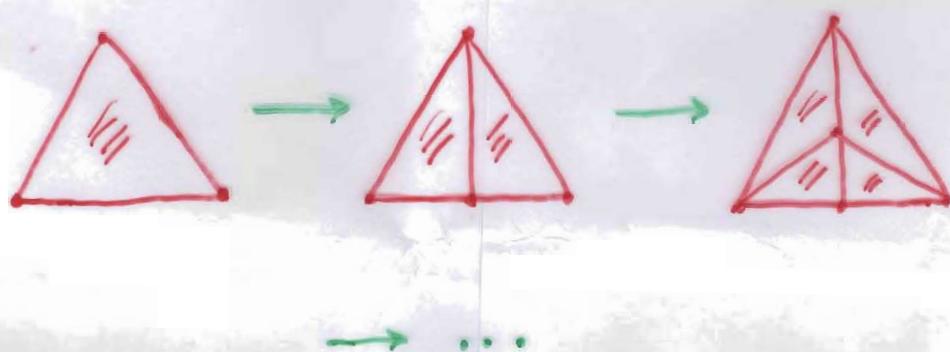
$$\begin{aligned} \gamma(\Delta', x) &= \gamma(\Delta, x) + \\ &\quad \sum_{F \in \Delta - \{\emptyset\}} \xi_F(\Delta'_F, x) \gamma(LK_\Delta(F), x). \end{aligned}$$

Conjecture 3 (A, 2012) $\mathfrak{F}_V(\Gamma, x) \geq 0$
(meaning, $e_V(\Gamma, x)$ is γ -nonnegative)
for every flag triangulation Γ of
the simplex Δ^V .

Theorem (A, 2012) This conjecture
(suitably generalized) implies both
Gal's conjecture and the Postnikov
- Reiner - Williams conjecture.

Theorem (A, 2012) Conjecture 3 holds:

- in dimension ≤ 3
- for successive edge subdivisions
of the trivial subdivisions of \mathbb{Q}^V

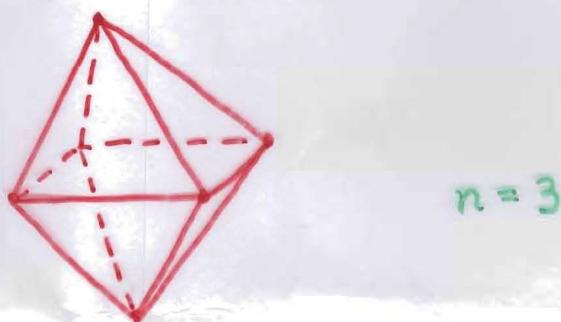


- for various examples discussed
in part D.

Open For barycentric subdivisions of
polyhedral subdivisions (or just trian-
gulations) of \mathbb{Q}^V .

why does Conjecture 3 imply Gal's conjecture? Let

Σ_{n-1} = boundary complex of
 n -dimensional cross-polytope



Theorem (A, 2012) Every flag triangulation Δ of S^{n-1} is a flag "vertex-induced, homology" subdivision Γ of Σ_{n-1} . Moreover,

$$g(\Delta, x) = \sum_{F \in \Sigma_{n-1}} \mathbb{E}_F(\Gamma_F, x).$$

D. Examples

Let

$$\begin{cases} V = n\text{-element set} \\ \Gamma = \text{triangulation of } \Delta^V \end{cases}$$

and recall that

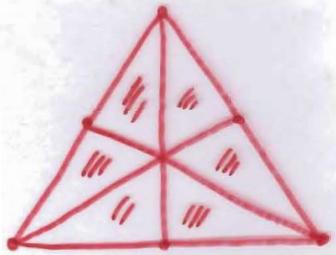
$$\begin{cases} \ell_V(\Gamma, x) = \sum_{i=0}^n \ell_i(\Gamma) x^i \\ \xi_V(\Gamma, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Gamma) x^i \end{cases}$$

where

$$\begin{aligned} \ell_V(\Gamma, x) &= \sum_{F \subseteq V} (-1)^{|F|} h(\Gamma_F, x) \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Gamma) x^i (1+x)^{n-2i}. \end{aligned}$$

Problem For nice combinatorially defined examples of flag triangulations interpret $\epsilon_i(\Gamma)$ and $\delta_i(\Gamma)$ combinatorially and verify that $\delta_i(\Gamma) \geq 0$.

Let



$$n = 3$$

$$\Gamma = sd(2^{\vee})$$

= barycentric sub-
division of 2^{\vee}

Proposition (Stanley, 1992)

$$\ell_{\nabla}(\Gamma, x) = d_n(x)$$
$$:= \sum_{w \in D_n} x^{\text{exc}(w)}$$

where D_n is the set of derangements
in S_n and $\text{exc}(w) = \#\{1 \leq i \leq n : w(i) > i\}$
(is the number of excedances of w).

Remark The derangement polynomial $d_n(x)$ was studied by

- Brenti (1990)
- Stembridge (1992)
- Zhang (1995)
- ...

Example For $n=4$

$$\begin{aligned}d_4(x) &= x + 7x^2 + x^3 \\&= x(1+x)^2 + 5x^2\end{aligned}$$

so

$$\xi_V(\Gamma, x) = x + 5x^2.$$

Theorem (A-Savvidou, Gessel-Shareshian
-Wacks, Shin-Zeng, Linusson-Shareshian
-Wacks)

$$d_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} S_i x^i (1+x)^{n-2i}$$

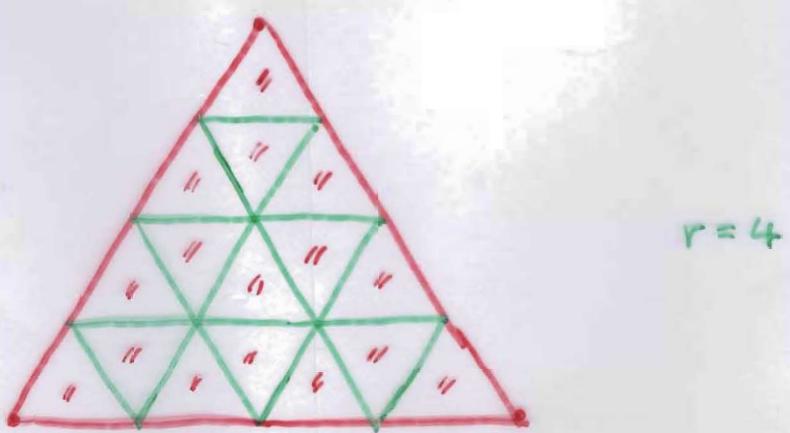
where S_i is the number of permutations $w \in G_n$ with i ascending runs and no ascending run of length one.

Example $n=4$

- 1234
- 13 • 24
- 14 • 23
- 24 • 13
- 23 • 14
- 34 • 12

so $S_1 = 1, S_2 = 5$.

Let Γ be the r th edgewise subdivision
of 2^V



studied by

- Freudenthal (1992)
- Grayson (1989)
- Edelsbrunner - Grayson (2000)
- Brun - Römer (2005)
- Brenti - Welker (2009)
- Kubitzke - Welker (2012)

Theorem (A, 2013)

$$e_V(\Gamma, x) = \sum_w x^{\text{asc}(w)}$$

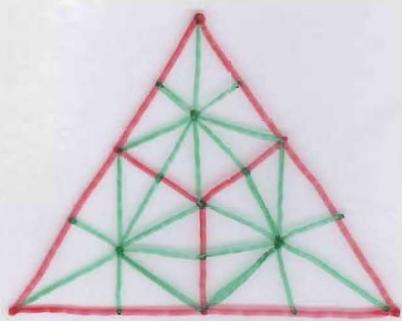
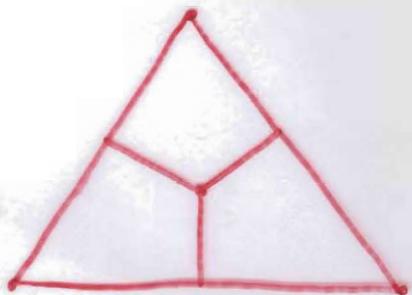
where $w = (w_1, w_2, \dots, w_{n-1})$ runs through all elements (Smirnov words) of the set $\{0, 1, \dots, r-1\}^{n-1}$ satisfying $w_i \neq w_{i+1}$ for $0 \leq i \leq n-1$, with $w_0 = w_n = 0$, and

$$\text{asc}(w) = \# \{0 \leq i \leq n-1 : w_i < w_{i+1}\}.$$

Moreover, $E_V(\Gamma, x) \geq 0$.

Let

- K_n = barycentric subdivision of cubical barycentric subdivision of 2^V .



Remark

$$\begin{aligned} \ell_V(K_n, 1) &= \# \text{ derangements in} \\ &\quad \text{Alt}(B_n) \\ &= \# \text{ derangements in } D_n \end{aligned}$$

where D_n is the group of even signed permutations.

Let $w = (w_1, w_2, \dots, w_n) \in \mathbb{B}_n$.

Definition (Bagno - Garber, 2006) The flag excedance of w is defined as

$$fex(w) = 2 \cdot \text{exc}_A(w) + \text{neg}(w)$$

where $\text{exc}_A(w) = \#\{1 \leq i \leq n : w_i > i\}$ and $\text{neg}(w)$ is the number of negative coordinates of w .

Example For $w = (3, -5, 1, 4, -2)$ we have $\text{exc}_A(w) = 1$ and $\text{neg}(w) = 2$, so $fex(w) = 4$.

Theorem (A, 2013)

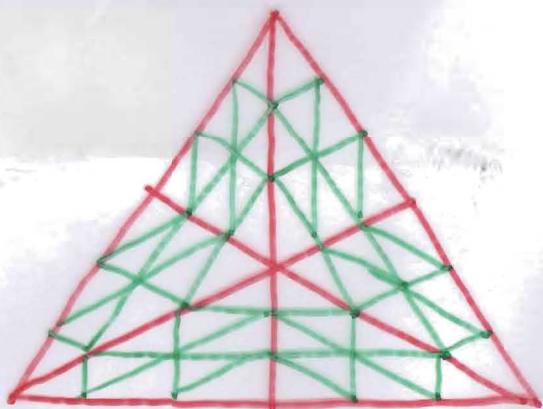
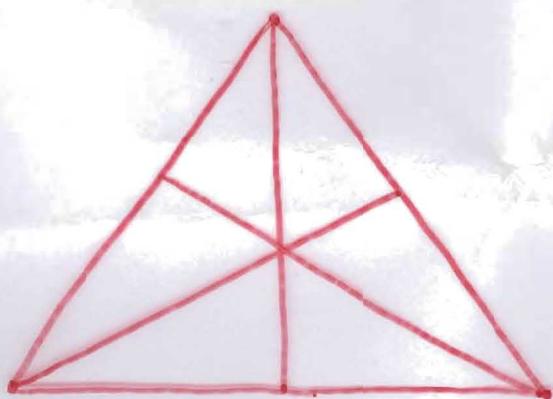
$$\ell_{\nabla}(K_n, x) = \sum_w x^{\text{tex}(w)/2}$$
$$= \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,i}^+ x^{i(n-2i)},$$

where the first sum ranges over all derangements $w \in D_n \subseteq B_n$ and $\xi_{n,i}^+$ is the number of elements of B_n with i decreasing runs, none of size one and positive last coordinate.

This can be generalized to the group
of r -colored permutations $\mathbb{Z}_r \wr G_n$:

Let

$\Gamma_n(r)$ = r th edgewise subdivision
of barycentric subdivi-
sion of Δ^n .



$$n=r=3$$

Proposition $e_{\Delta^n}(K_n, x) = e_{\Delta^n}(\Gamma_n(x), x)$.

For $w \in \mathbb{Z}_r \wr G_n$ set

$$fex(w) = r \cdot \underset{\Delta}{\text{exc}}(w) + csum(w)$$

where $csum(w)$ is the sum of the colors of all coordinates of w and call w balanced if $csum(w) \equiv 0 \pmod{r}$.

Theorem (A, 2013)

$$\begin{aligned} e_V(\Gamma_n(r), x) &= \sum_w x^{fex(w)/r} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,r,i}^+ x^i (1+x)^{n-2i}, \end{aligned}$$

where w ranges over all balanced derangements in $\mathbb{Z}_r \wr G_n$ and $\xi_{n,r,i}^+$ is the number of elements of $\mathbb{Z}_r \wr G_n$ with i decreasing runs, none of size one, and last coordinate of zero color.